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The origin of mass from Extra Dimensions

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Introducción

El Modelo Estandard de las interacciones fundamentales (MS) representa la mejor teoría que tenemos a nuestra disposición para describir los procesos de alta energía. A pesar de este hecho, esta teoría no puede ser la descripción definitiva de la Naturaleza. De hecho, el MS todavía contiene muchos aspectos que no se entienden completamente, así como problemas sin resolver. En la primera categoría podemos citar el *puzzle* del sabor, el problema fuerte de CP o el problema de las jerarquías, mientras que en la segunda categoría se encuentran por ejemplo, la explicación de bariogénesis, el origen de las masas de los neutrinos, la formulación de la gravedad cuántica o la justificación de la constante cosmológica. En esta tesis centraremos nuestra atención principalmente en el *puzzle del sabor* y el *problema de la jerarquía*.

En el MS, los parámetros de sabor (masas de quarks y leptones, ángulos de mezcla y fases de violación de CP) son parámetros renormalizables, fijados mediante la comparación con los datos experimentales. Los quarks y los leptones cargados presentan una jerarquía de masas, mientras que las mezclas son muy diferentes en los sectores hadrónico y leptónico. El MS no aporta ninguna explicación para esa jerarquía y esa estructura. A diferencia de los acoplos entre fermiones y partículas de espín uno, los cuales se entienden a la perfección a traves de las interacciones de Yang-Mills, los acoplos de Yukawa todavía no han alcanzado semejante nivel de comprensión. Las réplicas por familias y los patrones de masas y mezclas constituyen el llamado *puzzle* del sabor. Podría resultar una pista de posible física más allá del MS, donde el origen de los acoplos de Yukawa y la jerarquía de los parámetros de sabor pudiesen encontrar una explicación natural.

En la literatura, el acercamiento normal al problema asume la existencia de una simetría nueva que prohibe (algunos) de los acoplos de masas del MS: La jerarquía de las masas y las mezclas (y las fases de violacion CP si corresponde) están parametrizadas en términos de operadores de mayor dimensión que aportan interacciones de Yukawa efectivas con la intensidad apropiada. Estas nuevas simetrías que intervienen en el espacio del sabor se denominan simetrías horizontales, en contraste con las simetrías (verticales) de las teorías de gran Unificación (GUT).

Desafortunadamente (o no), no hay una única forma de escoger una simetría de sabor que dé lugar a resultados fenomenológicamente aceptables. De hecho, muchos intentos considerando simetrías de sabor con diferentes características (discretas [1, 2] o continuas

[3–8], locales [7,8] o globales [3–6], abelianas [6–8] o no-abelianas [3–5]) pueden encontrarse en la literatura. Una de las primeras y por otro lado la más famosa entre todas las ideas es la que tuvieron Froggatt y Nielsen [6]. Clasificaron la "intensidad" de los acoplos de Yukawa efectivos en función de una carga $U(1)$ desconocida. El resultado es que el nivel de supresión de un determinado acoplo de Yukawa esta relacionado a su carga de Froggatt-Nielsen.

El problema de las jerarquías, a diferencia del problema anterior, está relacionado con el único escalar fundamental que aparece en el MS: El bosón de Higgs. Los datos experimentales indican que la masa del bosón de Higgs es del orden de la escala electrodébil. Esta masa es demasiado ligera si existe nueva física a una escala mayor, a la que el bosón de Higgs es sensible.

A diferencia de las masas de los bosones gauge y los fermiones, el término de masa del Higgs en el Lagrangiano del MS es una cantidad invariante gauge, y por lo tanto, no está protegida de adquirir valores altos por la simetría gauge. La naturaleza del problema de las jerarquías se entiende mejor si se considera el MS como una teoría efectiva válida hasta escalas de energía de orden Λ, por encima de las cuales debe reemplazarse por otra teoría (todavía desconocida) microscópica y fundamental. A nivel cuántico, la masa del Higgs depende tremendamente en los detalles de dicha teoría microscópica. Por ejemplo, usando una regularización simple con un *cut-off* a la escala Λ, nos encontramos que la masa del Higgs adquiere correcciones radiativas que dependen cuadráticamente de Λ. El valor exacto de Λ no se conoce, pero el éxito fenomenológico del MS pone una cota en dicho valor: $\Lambda \geq$ unos pocos TeV (ver *e.g.* ref. [9]). La escala Λ puede ser incluso tan grande como aquella en la que aparecen los efectos de la gravedad cuántica, la escala de Planck M_{Planck}. Este es el denominado "gran problema de la jerarquía", es decir, por qué las cotas experimentales indican una masa de Higgs del orden $\mathcal{O}(100GeV) << M_{Planck}$. Incluso en el caso de que nueva física ya apareciese al valor mínimo permitido, $\Lambda \sim$ TeV, se mantendría el problema de por qué y cómo la escala electrobébil (y por tanto la masa del Higgs) se estabiliza a un valor que es aproximadamente un orden de magnitud inferior a Λ. Algunas veces nos referimos a este último problema como el "pequeño problema de la jerarquía".

Se han propuesto muchas posibles soluciones al problema de la jerarquía gauge y se han explorado diferentes caminos para proteger la masa de Higgs de correcciones ultravioleta:

- Higgs como un supercompañero de un fermión (*supersimetría*).
- Higgs como un bosón de Goldstone de una simetría global espontáneamente rota (*tecnicolor* y *little Higgs*).
- Higgs como un componente de un bosón gauge (*dimensiones extra*).

Independientemente de la naturaleza precisa del campo de Higgs que se asume en cada una de las propuestas anteriores, todas requieren, de una forma u otra, la aparición de nueva física a $\Lambda \sim$ TeV.

El Modelo Estandard Supersimétrico Mínimo (MSSM de sus siglas en inglés) es, por el momento, el mejor candidato de nueva física más allá del MS. Sin embargo, no se ha descubierto aún ninguna partícula supersimétrica y el MSSM necesita cierto ajuste poco natural y no deseado de los parametros [10] cuando se compara con los datos obtenidos por LEP, medidas de $g-2$, desintegraciones extrañas o momentos dipolares eléctricos. Además, hay que prestar cierta atención para evitar la aparición de masas demasiado ligeras para los bosones de Higgs del MSSM. Por lo tanto, por todo lo citado anteriormente, resulta de vital importancia investigar soluciones alternativas al problema de las jerarquías.

En Tecnicolor [11] y escenarios de little Higgs [12–15], el Higgs del MS se identifica con el bosón de Goldstone de una simetría global espontáneamente rota. Las propuestas tradicionales de tecnicolor a la escala del TeV entran en serio conflicto con la fenomenología [16], dado que en general inducen contribuciones a las llamadas "correcciones oblicuas", que no son compatibles con las cotas experimentales. Los modelos de little Higgs reintroducen esta idea básica. Aquí, la simetría global está (parcial y) explicitamente rota por acoplos gauge. En contraposición a los antiguos modelos de tecnicolor, como mínimo dos acoplos contribuyen a la masas de Higgs, que es suficiente para garantizar que no es sensible a divergencias cuadráticas a un *loop*. Los modelos concretos, sin embargo, son algo artificiales y cuando se analizan en detalle, se ven tambien afectados por una necesidad poco deseable de ajustar los parametros hasta un nivel poco natural (el denominado *fine-tuning*) [17].

Una posibilidad diferente es considerar teorías formuladas en $D > 4$ dimensiones espacio-temporales. Existen varios marcos teóricos en el contexto de dimensiones extra. En esta tesis, nos centramos en la idea de que el bosón de Higgs del MS pueda surgir de componentes internas (es decir, de componentes espaciales extra) de un campo gauge de un grupo $G \supset G_{SM} \equiv SU(3)_c \times SU(2)_L \times U(1)_Y$ de más dimensiones [18–21]. Eligiendo grupos gauge apropiados en las dimensiones extra, se pueden incluir todos los bosones gauge del MS (γ, $W^{\pm}$, Z y gluones) así como al campo de Higgs H provinientes de diferentes componentes del mismo campo gauge en más dimensiones A_M, con M variando sobre todas las (habituales y extra) coordenadas espacio-tiempo.

Debido a este origen común de los campos gauge y de Higgs, este marco se denomina algunas veces "unificación gauge-Higgs" [22–43]. Su característica esencial es que, siendo el campo de Higgs una componente del campo gauge, la simetría gauge de más dimensiones subyacente protege su masa de divergencias radiativas cuadráticas.

Una construcción de modelos realista necesita incluir dos ingredientes fundamentales presentes en el MS: la presencia de fermiones quirales y la implementación de la ruptura de simetría electrodébil.

Quiralidad

En el MS, las dos componentes quirales de un fermión se comportan de diferente forma con respecto a la interacción electrodébil. Por el mismo hecho, las masas de Dirac no son invariantes gauge y los fermiones (excepto quizás los neutrinos) deben permanecer sin masa hasta que el mecanismo de ruptura de la simetría electrodébil esté operativo. Para poder obtener modelos efectivos de 4 dimensiones realistas, tenemos que ser capaces de producir fermiones con $D = 4$, con números cuánticos que varían con la quiralidad, provenientes de campos de mayores dimensiones. Un fermión viviendo en un espacio-tiempo plano de más dimensiones, puede descomponerse siempre en un número igual de $D = 4$ fermiones quirales levógiros y dextrógiros degenerados en energía. Esta es una consecuencia directa de la invariancia Poincaré de D-dimensiones. Estos $D = 4$ fermiones quirales levógiros y dextrógiros son componentes del mismo campo con dimensiones extra y, por lo tanto, tienen los mismos números cuánticos gauge. Por esta razón, los primeros intentos de compactificar las dimensiones extra atraviesan por la dificultad de obtener modelos quirales [44]. Dicho problema se presenta incluso partiendo de fermiones con más dimensiones que son quirales con respecto al espacio total.

Afortunadamente, las dos quiralidades $D = 4$ tienen diferentes comportamientos bajo simetrías geométricas en dimensiones extra (invariancia rotacional y de forma eventual, paridad). Este hecho permite solucionar el problema de la quiralidad. Si el mecanismo de compactificación rompe (*i.e.* fija) todas las simetrías de las dimensiones extra, se conseguirá siempre una teoría efectiva en cuatro dimensiones con un número distinto de $D = 4$ fermiones levógiros y dextrógiros con los mismos números cuánticos. En particular, si el mecanismo de compactificación es capaz de romper algunas de las simetrías gauge de D-dimensiones, realmente se obtienen fermiones en 4-dimensiones con números cuánticos que varían con la quiralidad en $D = 4$, comenzando con, únicamente, un campo con dimensiones extra. Dos mecanismos principales se usan en la literatura para implementarlo: *compactificación en orbifold* y *compactificación con background*.

Compactificación en orbifolds: Compactificación de las dimensiones extra en variedades planas con puntos singulares [45, 46].

En este tipo de compactificación, los componentes quirales izquierda y derecha se pueden elegir tales que se comporten de forma diferente en los puntos singulares, e incluso algunas componentes pueden desaparecer. Esto se consigue a través de una elección apropiada de las condiciones de contorno. Dichas condiciones de contorno tienen, necesariamente, que romper todas las simetrías geométricas de las dimensiones extra de tal forma que den lugar a $D = 4$ fermiones levógiros y dextrógiros, no-degenerados en puntos fijos.

Compactificación con background: compactificación considerando espacios en los cuales un campo de fondo (*background*) está presente. Dos tipos de *background* se han considerado:

background escalar de D-dimensiones, normalmente denominado escenario *domain wall* [47], y *backgrounds*s gauge (y a veces de gravedad) con una intensidad de campo no trivial: el llamado *flux compactification* [18–21, 48–53].

Nos concentraremos en el gauge flux compactification: compactificación en presencia de un *background* gauge con intensidad de campo constante, dando lugar a un espacio tiempo no-singular y suave. Dicho tipo de *background* es denominado *background magnético.* La presencia de un *background* magnético rompe todas las simetrías geométricas de las dimensiones extra dando lugar a fermiones quirales. La quiralidad obtenida de esta manera puede verse como una ruptura de degeneración hiperfina: las diferencia entre las masas de las dos quiralidades aparece y es proporcional a la intensidad del campo del *background* estable.

La idea de obtener fermiones quirales mediante compactificación en presencia de *backgrounds*s gauge abelianos y gravitacionales fue ilustrada por primera vez por S. Randjbar-Daemi, Abdus Salam and J.A. Strathdee [48], en un espacio-tiempo de 6 dimensiones con las dos dimensiones espaciales extra compactificadas en una esfera. Esta idea seminal de un *background* magnético dando lugar a la quiralidad para fermiones, se retomó de nuevo en el contexto de teoría de cuerdas, más concretamente con la intención de obtener fermiones quirales en construcciones de la cuerda heterótica [54].

Ruptura de simetría

La implementación de ruptura de simetría en el contexto de las dimensiones extra es otro ingrediente fundamental para la construcción de modelos realistas. Es interesante, en particular, entender si se puede imitar el mecanismo de Higgs estándar, sin estar afectado por el problema de la jerarquía electrodébil. Este punto representa el corazón central de la tesis. Analizaremos este problema en el contexto de *orbifolds* y compactificación de flujos.

Compactificación en orbifolds: En este caso, el problema relacionado a la ruptura de simetría ha sido analizado profundamente antes y en la literatura se pueden encontrar (al menos para el caso de una y dos dimensiones extra) recetas sobre herramientas e ingredientes que pueden usarse para una construcción de modelos realistas [55–57].

En la compactificación en *orbifolds*, las mismas condiciones de contorno capaces de inducir quiralidad pueden usarse para inducir ruptura de simetría gauge: esto se alcanza gracias a condiciones de contorno que actúan de forma diferente sobre componentes diferentes de los bosones gauge que viven en las dimensiones extra. El mecanismo de ruptura de simetría mediante *orbifolds* es un mecanismo de ruptura de simetría explícita en los puntos fijos. El carácter fundamentalmente local de este tipo de ruptura de simetría hace la teoría sensible a detalles ultravioletas, lo cual no es adecuado para nuestro propósito de solucionar el problema de las jerarquías. Necesitamos acoplar dicha construcción con un mecanismo de ruptura de simetría diferente.

La compactificación en una variedad no-simplemente conexa, ofrece otra posible implementación de la ruptura de simetría: el mecanismo de *Scherk-Schwarz* (SS) [58, 59]. Consiste en especificar condiciones de periodicidad no-triviales alrededor de caminos no-contráctiles de una variedad no-simplemente conexa. La característica no-local de esto último lo hace insensible a la dinámica local, es decir, a las divergencias ultravioletas. El mecanismo SS rompe la simetría gauge espontáneamente [60–62]. Desde el punto de vista de construcción de modelos, por lo tanto, el mecanismo SS podría ser una alternativa al mecanismo de Higgs. Resumiendo, el mecanismo SS es necesario para una ruptura de simetría gauge no-local, mientras que los *orbifolds* se requieren para obtener quiralidad.

Sin embargo, la compactificación en *orbifolds* es siempre delicada dado que puede permitir operadores localizados en puntos singulares, los cuales pueden no ser invariantes respecto a todas las simetrías del *bulk*. Como resulta evidente, si las simetrías en los puntos singulares permiten un término de masa para las componentes de bosones gauge de D-dimensiones que juegan el papel del bosón de Higgs del Modelo Estándar, esta masa resultaría infectada por correcciones radiativas divergentes (tal y como ocurre en el MS).

En la literatura [22, 27, 31, 34, 35, 37, 39–43, 63–65] el caso más discutido ha sido $D = 5$ debido a que en este caso particular, el parámetro de ruptura de simetría SS es insensible a la dinámica local, es decir, a divergencias ultravioletas [66]. Sin embargo, modelos fenomenológicos en $D = 5$ sufren del hecho de que la masa de Higgs tiende a ser demasiado baja, porque el potencial de Higgs de 4 dimensiones es completamente radiativo y por construcción, no existen términos cuárticos a nivel árbol [1] [22].

El caso $D = 6$, por contra, aparecería como un escenario prometedor ya que una contribución al potencial escalar $D = 4$ está ya presente a nivel árbol. Sin embargo, en este caso, no existe ninguna simetría residual no-lineal capaz de proteger la masa de Higgs de una correccion radiativa divergente localizada [28, 32, 67].

La perspectiva de resolver el problema de las jerarquías ha sido, sin lugar a dudas, la principal motivación para llevar a cabo el estudio de modelos en dimensiones extra. Resulta interesante, por otro lado, entender si es posible resolver otros problemas típicos del MS en el mismo contexto. En la parte final de esta tesis, presentaremos un modelo de sabor en el contexto de cinco dimensiones, con la dimensión extra compactificada en un *orbifold.* Mostraremos cómo una simetría abeliana de sabor à la Froggatt-Nielsen se puede incorporar de manera natural en modelos con unificación gauge-Higgs, mediante la explotación de los fermiones pesados, que son indispensables a la hora de obtener acoplos de Yukawa realistas. El caso del modelo de cinco dimensiones mínimo, en el cual el grupo electrodébil $SU(2)_L \times U(1)_Y$ se agranda a un grupo $SU(3)_W$, y después se rompe a un $U(1)_{\text{em}}$ mediante la combinación de una proyección de *orbifold* y una condición de periodicidad de Scherk-Schwarz, se estudia en detalle. Mostramos que la forma mínima

[1] El origen putativo de dichos términos sería las componentes de dimensiones extra de la intensidad de campo gauge, F_{ij}, que no esta presente en $D = 5$.

de incorporar una simetría de sabor $U(1)_F$ es aumentarla a un grupo $SU(2)_F$, que se rompe entonces completamente mediante la misma proyección de *orbifold* y condiciones de periodicidad de Scherk-Schwarz. Las características generales de esta construcción, donde los fermiones ordinarios viven en branas definidas por los puntos fijos del *orbifold* y fermiones mensajeros viven en el *bulk*, son comparadas a aquellas de modelos de sabor ordinarios de cuatro dimensiones, y algunos ejemplos concretos serán construidos.

Compactificación de flujo: La literatura sobre la fenomenología de las compactificaciones de flujo no es tan abundante como en el caso del *orbifold*. Un análisis preliminar de posibles patrones de ruptura de simetría que puedan obtenerse, de forma compatible con la presencia de un *background* magnético, es por lo tanto necesario. La parte principal del trabajo original presentado en esta tesis está relacionado con este tema.

En particular, reconsideramos la idea de compactificación de flujo (gauge) en un espacio simple: seis dimensiones con las dos extra compactificadas en un dos-toro, $\mathcal{T}^2$. En dicho espacio, consideramos sólo *backgrounds* gauge con intensidad de campo constante, necesaria para obtener quiralidad en cuatro dimensiones.

Para grupos simplemente conexos tales como $SU(N)$, todas las configuraciones de *background* estables en un dos toro tienen intensidad de campo cero. Para estos grupos, por lo tanto, la quiralidad está prohibida. Para poder obtener modelos quirales, lo más sencillo consiste, por tanto, en aumentar el grupo gauge considerando una teoría gauge $U(N)$ en un espacio tiempo $\mathcal{M}_4 \times \mathcal{T}^2$. $\mathcal{M}_4$ denota el espacio ordinario de 4-dimensiones de Minkoski. En este caso, de hecho, el subgrupo abeliano (no-simplemente conexo) $U(1) \subset U(N)$ admite configuraciones de intensidad de campo estable y diferente de cero.

Así como inducir quiralidad, la presencia de un *background* magnético estable asociado al subrupo abeliano $U(1) \subset U(N)$ tiene otras consecuencias importantes: afecta al subgrupo no-abeliano $SU(N) \subset U(N)$, dando lugar a un *flujo no-abeliano de t' Hooft* no trivial [68]. Un flujo no-trivial no-abeliano de 't Hooft siempre induce cotas no-triviales a las condiciones de periodicidad SS de $SU(N)$ alrededor de los caminos no contráctiles de $\mathcal{T}^2$. Nos referiremos a estas cotas como *las condiciones de consistencia de 't Hooft*, y a las condiciones de periodicidad compatibles con ellas como condiciones de periodicidad SS *generalizadas*, dado que, en general, pueden depender de las coordenadas.

Nuestro propósito es, por lo tanto, determinar las consecuencias de la presencia de un flujo no-trivial no-abeliano de 't Hooft: *i.e.* determinar la energía del vacío, el número y las características de los vacíos, las simetrías residuales y la estabilidad cuántica del patrón de ruptura de simetría que es compatible con la presencia de condiciones de SS generalizadas.

Las condiciones de consistencia de 't Hooft admiten dos clases de soluciones: condiciones de periodicidad $SU(N)$ dependientes de las coordenadas [69] y condiciones de periodicidad $SU(N)$ constantes [70].

En el caso de condiciones de contorno dependientes de las coordenadas, el cálculo analítico de configuraciones de vacío estable compatibles con las condiciones de periodicidad, no es un tema trivial. Sin embargo, determinar configuraciones estacionarias (no necesariamente estables) resulta ser tarea más sencilla. Para poder encontrar el vacío verdadero, es posible, por tanto, expandir el sistema alrededor de un tipo de configuración estacionaria (calculable) y determinar entonces su estabilidad, analizando el potencial efectivo para los campos de fluctuación.

Por ejemplo, es posible considerar un *background* gauge $SU(N)$ que sea compatible con condiciones de periodicidad dependientes de las coordenadas y que sea a su vez una solución de las ecuaciones de movimiento con intensidad de campo constante. Este último se supone apuntando a lo largo de una dirección fija de la representación adjunta. Este *background* es, necesariamente, una función del flujo no-abeliano de 't Hooft (como demostraremos) e imita un *background* magnético. A pesar de todo, no coincide necesariamente con el mínimo de la acción. En este caso, puede dar lugar a la presencia de grados de libertad taquiónicos.

Un ejemplo histórico en teoría de campos de grados de libertad taquiónicos de componentes de campo de gauge no-abelianos es la denominada *inestabilidad de Nielsen-Olesen* [71–73]. Ellos estudiaron un escenario solamente con las cuatro dimensiones usuales, con la intención de justificar el confinamiento en QCD. Se consideró una teoría gauge $SU(2)$ en cuatro dimensiones con un *background* con intensidad constante, que vivía sólo en dos de ellas y apuntaba a una dirección fija en la representación adjunta. Nielsen y Olesen encontraron que aparecía una teoría efectiva en 2 dimensiones $U(1) \subset SU(2)$ invariante, incluyendo un potencial escalar con campos neutros y cargados. El primero corresponde a los niveles de Landau mientras que el segundo a los modos de Kaluza Klein. En ausencia del citado *background*, los dos "escalares" más ligeros deberían estar degenerados. En su presencia, una ruptura de niveles de energía hiperfina aparece de forma automática, y de esta forma, estos dos escalares adquieren masas al cuadrado que son opuestas en signo. Una de las masas es taquiónica y por lo tanto, debe inducir ruptura de simetría espontánea "gratis": la simetría $U(1)$ debe estar presente pero escondida. Este fenómeno se denomina en la literatura *inestabilidad de Nielsen-Olesen*. El significado del *background* y la subsiguiente inestabilidad, en el contexto de cuatro dimensiones infinitas, continúa siendo un problema controvertido en la literatura [74–76].

El primer resultado novedoso (en el marco de la compactificacióon de flujo) presentado en esta tesis es la solucióon de la *inestabilidad de Nielsen-Olesen* para una teoría gauge $SU(N)$ en $\mathcal{M}_4 \times \mathcal{T}^2$. Más en detalle, analizamos la ruptura de simetría inducida por la presencia de un *background* en el toro, el cual tiene intensidad constante y compatible con condiciones de periodicidad dependientes de las coordenadas. Resulta además emocionante considerar si el mecanismo de Nielsen-Olesen puede implementarse para los propósitos de la ruptura de simetría electrodébil. En lugar de agrandar el sistema con la intención de cancelar *ab initio* cualquier posible término taquiónico [77], exploramos cómo alcanzar un vacío estable a partir de una configuración inicial y determinamos las simetrías que se mantienen.

Nótese que resolver este tipo de problemas es equivalente al propósito anunciado con anterioridad, es decir, determinar el vacío y las simetrías residuales compatibles con soluciones de las condiciones de consistencia de 't Hooft, dependientes de las coordenadas.

Análisis explícitos en teoría de campos de los mínimos del Lagrangiano de 4 dimensiones efectivo en presencia de un *background* magnético, se han llevado a cabo en la literatura [71–73, 78] para $SU(2)$, a pesar de que se ha hecho de forma algo incompleta, debido a las dificultades técnicas asociadas a tratar de forma simultánea con Kaluza-Klein y niveles de Landau en interacción. Por otro lado, nosotros tendremos en cuenta el potencial efectivo $4D$ completo para el caso de $SU(2)$, incluyendo todos los términos de interacción trilineares y cuárticos. Esto requerirá encontrar un Lagrangiano de "gauge-fixing" apropiado cuando interactúan torres de Kaluza-Klein y niveles de Landau, una herramienta que no se ha desarrollado previamente en la literatura.

Además, será técnicamente necesaria la resolución de integrales involucrando dos, tres y cuatro Kaluza-Klein y niveles de Landau: Esto se llevará a cabo de forma analítica para todos los modos. En el presente caso, nos permitirán calcular el potencial en cuatro dimensiones, encontrar sus mínimos y determinar entonces sus simetrías y espectro. Estos resultados técnicos podrían ser útiles en escenarios más generales a los considerados en el presente trabajo. Por ejemplo, se ha sugerido que configuraciones inestables de flujo se pueden asociar con configuraciones de branas intersecantes e inestables [79]. En este contexto, nuestra teoría de campos se puede ver como la aproximación clásica a una desintegración de D-branas a través de una condensación de taquiones pertenecientes al sector de cuerda abierta [80].

Siendo $SU(N)$ el grupo gauge de interés, el tratamiento de teoría de campos que se acaba de describir habría sido innecesario dado que argumentos teóricos puros permiten establecer las simetrías del vacío estable. Por otro lado, en $\mathcal{T}^2$, el *background* con intensidad constante en el origen de la inestabilidad de Nielsen-Olesen requiere condiciones de contorno para los campos dependientes de las coordenadas. Como probaremos y discutiremos en profundidad, para $SU(N)$ en un dos-toro, las condiciones de periodicidad dependendientes de las coordenadas son equivalentes gauge a las constantes. Además, todos los *background* $SU(N)$ estables en $\mathcal{T}^2$ tienen intensidad cero y por lo tanto, energía cero [81–83]. Este resultado nos permite demostrar que todos los *backgrounds* estables son equivalentes gauge al trivial. Las simetrías del espectro en 4 dimensiones, por tanto, se pueden inferir directamente analizando un sistema con condiciones de periodicidad constantes.

Las simetrías del vacío dependen esencialmente de si los flujos de 't Hooft presentes son triviales o no-triviales. Esto se traduce entonces en saber si las condiciones de contorno constantes corresponden a líneas de Wilson continuas o discretas. Mientras la mayoría de la literatura se dedica al caso de líneas de Wilson continuas, uno de los ingredientes nuevos de esta tesis es el análisis fenomenológico del patrón de la ruptura de simetría gauge y el espectro de excitaciones gauge y escalares en 4 dimensiones, para el caso general de $SU(N)$ y líneas de Wilson discretas. Los resultados se mostrarán consistentes con los obtenidos a partir del análisis de teoría de campos del Lagrangiano efectivo, para el caso de $SU(2)$, apoyando de esta forma la consistencia de la aproximación mediante teoría de campos desarrollada en este trabajo.

La última parte del trabajo sobre compactificacón de flujo se dedica al análisis de la estabilidad cuántica de la ruptura de simetría. El hecho de que para $U(N)$ (o equivalentemente $SU(N)$ con flujo 't Hooft no-abeliano y no-trivial) sea posible interpretar las configuraciones de vacío estable en términos de condiciones de periodicidad SS constantes, sugiere que dicha ruptura de simetría tenga una naturaleza no-local. Para entender y clarificar dicho punto, calcularemos explícitamente el potencial efectivo a un *loop* usando la técnica de Heat-Kernel. El cálculo de Heat Kernel, debido a que tiene lugar en el espacio de coordenadas, aparece como un instrumento muy útil para distinguir contribuciones provinientes de diagramas locales (sensibles al ultra-violeta) y no-locales (insensibles al ultra-violeta). Las contribuciones locales no dependen en las condiciones de periodicidad y son invariantes bajo todas las simetrías originales. No contribuyen a los parámetros de orden de la ruptura de simetría. Solamente las contribuciones no-locales serán relevantes para la ruptura de simetría, que está entonces protegida de las divergencias ultravioletas.

Estructura de la tesis

La tesis está organizada de la siguiente forma.

Los capítulos 1 y 2 aportan una introducción general a la teoría de campos en dimensiones extra compactificadas.

Repasamos el concepto de compactificación en manifolds de 1- y 2-dimensiones con y sin puntos fijos. Un papel importante lo juega la discusión sobre quiralidad en los dos marcos diferentes introducidos con anterioridad: compactificaciones en *orbifolds* (capítulo 1) y flujo (capítulo 2). Con respecto a compactificación en *orbifold*, además, repasamos los principales mecanismos de ruptura de simetría así como el estudio de posibles patrones de ruptura de simetría que se pueden alcanzar. Incluso, además de ser útil para un lector novel, los capítulos 1 y 2 resultan a la vez bastante técnicos. Por lo tanto, sugerimos al lector interesado en aplicaciones (de fenomenología), pasar directamente al capítulo 3.

Los capítulos 3, 4 y 5 representan el corazón central de la tesis y estan centrados en el estudio de la ruptura de simetría gauge en el contexto de compactificación de flujo para ambos flujos no-abelianos de 't Hooft, trivial y no-trivial.

Los capítulos 3 y 4 se complementan perfectamente. En el capítulo 3, buscamos el vacío estable y sus simetrías para una teoría gauge $SU(N)$ en un espacio-tiempo de 6 dimensiones donde las 2 dimensiones extra están compactificadas en un toro, usando la teoría de campos efectiva.

En el capítulo 4, adoptamos un método más teórico y probamos cómo se pueden entender en términos de condiciones de periodicidad constante el vacío estable y las simetrías residuales. Este resultado es válido incluso en el caso de un flujo no-abeliano de 't Hooft no-trivial. Además, probamos de forma explícita qué patrones de ruptura de simetría se pueden conseguir.

En el capítulo 5, por otro lado, analizamos la estabilidad de este mecanismo de ruptura de simetría respecto a correcciones cuánticas. En particular, mostraremos que su naturaleza no-local lo hace insensible a escalas de energía mayores que la de compactificación. Este capítulo incluye el cálculo del potencial a un *loop* usando la técnica de Heat-Kernel, de tal forma que se tengan en cuenta los efectos provenientes de operadores no-locales.

La última parte de la tesis se dedica al problema del sabor.

El capítulo 6 contiene un breve resumen del problema del sabor en el MS y del mecanismo de Froggatt-Nielsen. El capítulo 7 aporta nuestra implementación de un modelo de sabor en el contexto de dimensiones extra. Para terminar, en el capítulo 8 presentamos nuestras conclusiones.

Introduction

The Standard Model of fundamental interactions (SM) represents the best theory at our disposal to describe high-energy processes. Most likely, however, it cannot be the definitive description of Nature. The SM still contains, indeed, many non-completely understood aspects as well as some unsolved problems. In the first category, we can recall the flavour puzzle or the hierarchy problem, whereas in the second category fall for example the explanation of baryogenesis, the strong CP problem, the origin and nature of neutrino masses, the formulation of quantum gravity or the justification of the cosmological constant. In this thesis we will focus our attention mainly on the *flavour puzzle* and the *hierarchy problem*.

In the SM, the flavour parameters (quark and lepton masses, mixing angles and CP-violating phases) are renormalizable parameters, fixed by comparison with the experimental data. Quarks and charged leptons present a hierarchy of masses, while mixings are very different in the hadronic and leptonic sectors. The SM does not provide any explanation about that hierarchy and structure. Unlike the couplings of fermions to spin-one particles, which are well understood in terms of Yang-Mills interactions, Yukawa couplings still await such level of understanding. The family replication and the pattern of masses and mixings constitutes the flavour puzzle. It may be a hint for physics beyond the SM, where the origin of the Yukawa couplings and the hierarchy of the flavour parameters could find a natural explanation.

In the literature, the standard approach to the problem is to assume the existence of a new symmetry that forbids (some) SM mass couplings: the hierarchies of masses and mixings and eventually the CP-violating phases would result from the breaking of such flavour symmetries. As the latter act on flavour space they are called horizontal symmetries, in contrast with the (vertical) symmetries of Grand Unified Theories (GUT).

Unfortunately (or not), there is not an unique way of choosing a flavour symmetry giving rise to phenomenological acceptable results. Many attempts, indeed, considering flavour symmetries with different characteristics (discrete [1, 2] or continous [3–8], local [7,8] or global [3–6], abelian [6–8] or non-abelian [3–5]) can be found in the literature. One of the earliest and the most famous idea is that advocated by Froggatt and Nielsen [6]. They organize the "intensity" of the effective Yukawa couplings in terms of an unknown $U(1)$ charge. The result is that the suppression level of a particular Yukawa coupling is related to its Froggatt-Nielsen charge.

The hierarchy problem, instead, is related to the only fundamental scalar appearing in the SM: the Higgs boson. Data indicate that the mass of the Higgs boson is of the order of the electroweak scale.

Such a mass is unnaturally light if there is new physics at a higher scale, to which the Higgs boson is sensitive. Unlike gauge bosons and fermion masses, the Higgs mass term in the SM Lagrangian is a gauge invariant quantity, and thus it is not protected by the gauge symmetry from acquiring large values.

The nature of the hierarchy problem is best understood if one considers the SM as an effective field theory valid up to energy scales of order Λ, above which the theory has to be replaced by a more fundamental (and yet unknown) microscopic theory. At the quantum level, the Higgs mass heavily depends on the details of the microscopic theory. For instance, by using a simple cut-off regularization at the scale Λ, one finds that the Higgs mass gets radiative corrections which are quadratically dependent on Λ. The value of Λ is unknown, but the phenomenological success of the SM puts a bound on it: $\Lambda \geq$ few TeV (see *e.g.* ref. [9]). The scale Λ can be even as large as that at which gravity quantum effects appear, the Planck scale M_{Planck}. This is the so-called "big hierarchy problem", that is, why the experimental constraints indicate a Higgs mass of order $\mathcal{O}(100 GeV) << M_{Planck}$. Even in the case that new physics would already appear at the minimal experimentally allowed value, $\Lambda \sim$ TeV, it would remain the problem of why and how the electroweak scale (and thus the Higgs mass) is stabilized to a value which is roughly one order of magnitude smaller than Λ. Sometimes one refers to this latter problem as the "little hierarchy problem".

Many solutions have been proposed to address the gauge hierarchy problem and different approaches have been explored to protect the Higgs mass from ultraviolet corrections:

- Higgs as a superpartner of a fermion (*supersymmetry*).
- Higgs as a Goldstone boson of a spontaneously broken global symmetry (*technicolor* and *little Higgs*).
- Higgs as a component of a gauge boson (*extra dimensions*).

Independently of the precise nature of the Higgs field that is assumed in each of these proposals, all of them require, in one way or another, the appearance of new physics at $\Lambda \sim$ TeV.

The Minimal Supersymmetric Standard Model (MSSM) is at the moment the best candidate theory of new physics beyond the SM. However, no super-particle has been discovered yet and the MSSM needs some unwanted fine tuning [10] when compared with LEP data, $g-2$ measurements, rare decays or electric dipole moments. Moreover, some attention has to be paid to avoid too light masses for the MSSM Higgs bosons. It is thus important to investigate alternative solutions to the hierarchy problem.

In technicolor [11] and little Higgs [12–15] scenarios, the SM Higgs is identified with the Goldstone boson of a spontaneously broken global symmetry. Traditional technicolor proposals at the TeV scale run into serious phenomenological problems [16], as in general they induce contributions to the so-called "oblique corrections", not compatible with experimental bounds. Little Higgs models reintroduce this basic idea. Here, the global symmetry is (partially and) explicitly broken by gauge couplings. Contrary to old technicolor models, at least two couplings contribute to the Higgs mass, which is sufficient to guarantee that it is not sensitive to one loop quadratic divergences. Concrete models are rather contrived, though, and when analyzed in detail they are also afflicted by fine-tuning requirements [17].

A different possibility is to consider theories formulated in $D > 4$ space-time dimensions. There are several theoretical frameworks in the context of extra dimensions. In this thesis, we focus on the idea that the SM Higgs boson may arise from the internal components (that is, the extra spatial components) of a higher-dimensional gauge field of a group $G \supset G_{SM} \equiv SU(3)_c \times SU(2)_L \times U(1)_Y$ [18–21]. By choosing suitable gauge groups in the extra dimensions, one can incorporate all the SM gauge bosons (γ, $W^{\pm}$, Z and gluons) and the Higgs field H as arising from different components of the same higher dimensional gauge field A_M, with M running over all (usual and extra) space-time coordinates.

Due to this common origin of the gauge and the Higgs fields, this framework is sometimes called "gauge-Higgs unification" [22–43]. Its essential point is that, being the Higgs field a component of a gauge field, the underlying higher-dimensional gauge symmetry protects its mass from radiative quadratic divergences.

Realistic model-building needs to include two fundamental ingredients present in the SM: the presence of chiral fermions and the implementation of electroweak symmetry breaking.

Chirality

In the SM, the two chiral components of a fermion behave in a different way with respect to the electro-weak interaction. By the same token, Dirac masses are not gauge invariant and fermions must remain massless until the electroweak symmetry breaking mechanism is operative. In order to achieve realistic 4-dimensional effective models, we have to be able to produce, out of higher-dimensional fields, $D = 4$ fermions with quantum numbers which vary with chirality.

A fermion living on a higher dimensional flat space-time can be always decomposed in an equal number of degenerate left- and right-handed $D = 4$ chiral fermions. This is a direct consequence of D-dimensional Poincaré invariance. These left- and right-handed $D = 4$ chiral fermions are components of the same higher dimensional field and therefore they have the same gauge quantum numbers. For this reason, the first attempts to compactify extra dimensions ran into the difficulty of obtaining chiral models [44]. Such a problem is present even starting from higher dimensional fermions which are chiral with respect to the total space.

Fortunately, the two different $D = 4$ chiralities have different behaviour under geometrical extra-dimensional symmetries (rotational invariance and eventually parity). This fact allows to overcome the chirality problem. If the compactification mechanism breaks (*i.e.* fixes) all the geometrical extra-dimensional symmetries, it will indeed always result in an effective four-dimensional theory with a different number of left- and right-handed $D = 4$ fermions with the same quantum numbers. In particular, if the compactification mechanism is able to break some D-dimensional gauge symmetries, it really achieves four dimensional fermions with quantum numbers which vary with the $D = 4$ chirality, starting from only one higher dimensional field. Two main mechanisms are used in the literature to implement it: *compactification on orbifold* and *compactification with background*.

Compactification on orbifold: compactification of the extra dimensions on flat manifolds with singular points [45,46]. In this type of compactification, left and right chiral components can be chosen to behave differently at the singular points, or even some components may vanish there altogether. This is achieved through an appropriate choice of boundary conditions. Such boundary conditions necessarily have to break all geometrical symmetries of the extra dimensions, in order to give rise to non-degenerate left- and right-handed $D = 4$ fermions at the fixed points.

Compactification with background: compactification considering spaces on which a background field is present. Two types of backgrounds have been considered: D-dimensional scalar backgrounds, usually denominated *domain wall* scenarios [47], and gauge (and eventually gravity) backgrounds with non trivial field strengths: the so-called *flux compactification* [18–21, 48–53].

We will concentrate on gauge flux compactification: compactification in the presence of a gauge background with constant field strength, resulting in a non-singular and smooth space-time. Such type of background is called *magnetic background*. The presence of a magnetic background breaks all geometrical symmetries of the extra dimensions, giving rise to chiral fermions. The chirality so obtained can be seen as a hyperfine splitting: the mass splitting between the two chiralities is, indeed, proportional to the field strength of the stable background.

The idea of obtaining chiral fermions by compactification in the presence of abelian gauge and gravitational backgrounds was first illustrated by S. Randjbar-Daemi, Abdus Salam and J.A. Strathdee [48], on a 6-dimensional space-time, with the two extra spatial dimensions compactified on a sphere. This seminal idea of a magnetic background resulting in chirality for fermions was also retaken in string theory, more concretely as a means of obtaining chiral fermions in the heterotic string constructions [54].

Symmetry breaking

The implementation of symmetry breaking in the context of extra dimensions is another fundamental ingredient for a realistic model building. It is interesting, in particular, to understand whether it may allow to mimic the standard Higgs mechanism, without being afflicted by the electroweak hierarchy problem. This point represents the core of the thesis.

Orbifold compactification. In this case, the problem regarding the symmetry breaking has been analyzed in depth and the literature already provides (at least in the case of one and two extra dimensions) a receipt about tools and ingredients that can be used for a realistic model building [55–57].

In orbifold compactifications, the same boundary conditions able to induce chirality may be used to induce gauge symmetry breaking: this is achieved by boundary conditions acting differently on different components of the extra dimensional gauge bosons. The orbifold symmetry breaking mechanism is an explicit symmetry breaking mechanism, acting at the fixed points.

Compactification on a non-simply connected manifolds offers, in addition, another possible implementation of symmetry breaking: the *Scherk-Schwartz* (SS) mechanism [58, 59]. It consists in specifying non-trivial periodicity conditions around the non-contractible cycles of a non-simply connected manifold. The essential non-local character of the latter makes it insensitive to the local dynamics, that is, to ultraviolet divergences. The SS mechanism breaks the gauge symmetry spontaneously [60–62]. From the model building point of view, therefore, the SS mechanism may be an alternative to the Higgs mechanism. Summarizing, the SS mechanism is necessary for non-local gauge symmetry breaking, while orbifolding is required by chirality.

Orbifold compactification, however, is always delicate. The essentially local character of this symmetry breaking may allow new operators localized at the singular points, which could be non-invariant with respect to all the bulk symmetries. Obviously, if the symmetries at the singular points allow a mass term for the components of the D-dimensional gauge bosons which play the role of the SM Higgs, this mass will turn out to be plagued by divergent radiative corrections (as in the SM).

In the literature [22, 27, 31, 34, 35, 37, 39–43, 63–65] the $D = 5$ case has been mainly discussed, since in this particular case the SS symmetry breaking order parameter is insensitive to the local dynamics, that is to the ultraviolet divergences [66]. However, phenomenological models in $D = 5$ turn out to suffer from the shortcoming that the Higgs mass tends to be too low, because the 4-D Higgs potential is completely radiative and by construction no quartic terms are present at tree level [2] [22].

The $D = 6$ case, instead, should appear as a promising scenario since a contribution to the $D = 4$ scalar potential is already present at tree level. However, in this case, it does not exist any non-linear residual symmetry able to protect the Higgs mass from localized divergent radiative corrections [28, 32, 67].

The perspective of solving the hierarchy problem has been, with no doubts, the main motivation for undertaking the study of extra-dimensional models. It is interesting, however, to understand whether other typical SM problems could be addressed in the same context. In the final part of this thesis we describe an excursion in this direction: we present a flavour model in the context of five dimensions, with the extra one compactified on a orbifold. We show that an abelian flavour symmetry à la Froggatt-Nielsen can be naturally incorporated in models with gauge-Higgs unification, by exploiting the heavy fermions that are anyhow needed to realize realistic Yukawa couplings. The case of the minimal five-dimensional model, in which the $SU(2)_L \times U(1)_Y$ electroweak group is enlarged to an $SU(3)_W$ group, and then broken to $U(1)_{\rm em}$ by the combination of an orbifold projection and a Scherk-Schwarz periodicity condition, is studied in detail. We show that the minimal way of incorporating a $U(1)_F$ flavour symmetry is to enlarge it to an $SU(2)_F$ group, which is then completely broken by the same orbifold projection and Scherk-Schwarz periodicity conditions. The general features of this construction, where ordinary fermions live on the branes defined by the orbifold fixed-points and messenger fermions live on the bulk, are compared to those of ordinary four-dimensional flavour models, and some explicit examples are constructed.

Flux compactification: The literature about the phenomenology of flux compactification is not as rich as in the orbifold case. A preliminary analysis of possible symmetry breaking patterns that can be achieved, compatibly with the presence of a magnetic background, is then necessary. The main part of the original work presented in this thesis is related to this topic.

In particular, we re-consider the idea of (gauge) flux compactification in a simple space: six dimensions with the two extra ones compactified on a two-torus, $\mathcal{T}^2$. In such space, we consider only gauge backgrounds with constant field strength, necessary to obtain four-dimensional chirality.

For simply connected groups such as $SU(N)$, all stable background configurations on a two-torus have zero field strength. For these groups, therefore, chirality is precluded. In order to obtain chiral models, the simplest setting consists, therefore, in enlarging the gauge group and considering a $U(N)$ gauge theory on a $\mathcal{M}_4 \times \mathcal{T}^2$ space-time, where $\mathcal{M}_4$ denotes the ordinary four-dimensional Minkoski space. In this case, indeed, the abelian (non-simply connected) subgroup $U(1) \subset U(N)$ admits stable non-zero field strength

[2]The putative origin of such terms would be the extra-dimensional components of the gauge field strength, F_{ij}, absent in $D = 5$.

configurations.

As well as inducing chirality, the presence of a stable magnetic background associated with the abelian subgroup $U(1) \subset U(N)$ has other important consequences: it affects the non-abelian subgroup $SU(N) \subset U(N)$, giving rise to a non-trivial *t' Hooft non-abelian flux* [68]. A non-trivial 't Hooft non-abelian flux always induces non-trivial constraints to the $SU(N)$ SS periodicity conditions around the non-contractible cycles of $\mathcal{T}^2$. We will refer to these constraints as *'t Hooft consistency conditions* and to the periodicity conditions compatible with them as *generalized* SS periodicity conditions since, in general, they can depend on the coordinates.

Our aim is, therefore, to determine the consequences of the presence of non-trivial 't Hooft non-abelian flux: *i.e.* to determine the vacuum energy, the number and the characteristics of vacua, the residual symmetries and the quantum stability of the symmetry breaking pattern which are compatible with the presence of generalized SS periodicity conditions.

The 't Hooft consistency conditions admit two classes of solutions: $SU(N)$ coordinate-dependent periodicity conditions [69] and $SU(N)$ constant periodicity conditions [70].

In the case of coordinate-dependent boundary conditions, the analytical computation of stable vacuum configurations compatible with the periodicity conditions, is not a trivial issue. However, to determine stationary (not necessarily stable) configurations turn out to be an easier task. In order to find the true vacuum, it is possible, therefore, to expand the system around such a (computable) stationary configuration and to determine then its stability, analyzing the effective potential for the fluctuation fields.

For example, it is possible to consider a $SU(N)$ gauge background which is compatible with coordinate-dependent periodicity conditions and which is a solution of the equations of motion with constant field strength. The background is assumed to point along a fixed direction of the adjoint representation. It is necessarily a function of the 't Hooft non-abelian flux (as we will show) and mimics a magnetic background. Nevertheless, it does not necessarily coincide with a minimum of the action. In this case, it can give rise to the presence of tachyonic degrees of freedom.

A historical field theory example of tachyonic degrees of freedom stemming from components of non-abelian gauge fields is the so-called *Nielsen-Olesen instability* [71–73]. They studied a scenario within only the four usual flat dimensions, in order to justify confinement in QCD. A $SU(2)$ gauge theory in four dimensions was considered, with a background with constant field strength, that lived only on two of them and pointed to a fixed direction in the adjoint representation. They found that it resulted in an effective 2-dimensional $U(1) \subset SU(2)$ invariant theory, including a scalar potential with charged and neutral fields. The former correspond to Landau levels whereas the latter ones to Kaluza Klein modes. In the absence of such background, the two lightest charged "scalars" would be degenerate. In its presence, hyperfine splitting follows automatically, though, with those two scalars acquiring squared-masses which are opposite in sign. One of the masses is tachyonic and thus may induce spontaneous symmetry breaking "for free": the $U(1)$ symmetry may be there but hidden. Such phenomenon is called in the literature *Nielsen-Olesen instability.* The meaning of the background and the subsequent instability, in the context of four infinite dimensions, is still a very controversial problem in the literature [74–76].

The first novel result (in the framework of flux compactification) presented in this thesis is the solution of the *Nielsen-Olesen instability* for a $SU(N)$ gauge theory on $\mathcal{M}_4 \times \mathcal{T}^2$. More in detail, we analyze the symmetry breaking induced by the presence of a background on the torus, which has constant field strength compatible with coordinate-dependent periodicity conditions. It is indeed intriguing to consider whether the Nielsen-Olesen mechanism can be implemented for the purpose of electroweak symmetry breaking. Instead of enlarging the system so as to cancel *ab initio* any possible tachyonic term [77], we explore how a stable vacuum is reached from the initial configuration and we determine its remaining symmetries.

Notice that to solve such a problem is equivalent to the aim announced before, that is, to determine the vacua and residual symmetries compatible with coordinate-dependent solutions of the 't Hooft consistency conditions.

Explicit field theory analysis of the minima of the effective four-dimensional Lagrangian in the presence of backgrounds have been attempted in the literature [71–73, 78] for $SU(2)$, although in a rather incomplete way, due to the technical difficulties associated to handling simultaneously Kaluza-Klein modes and Landau levels in interaction. In contrast, we will take into account the complete effective $4D$ potential for the case of $SU(2)$, including all trilinear and quartic interaction terms. This will require to find a gauge-fixing Lagrangian appropriate when interacting towers of Kaluza-Klein modes and Landau levels are present, a tool not previously developed in the literature. Furthermore, it will be technically necessary to solve integrals involving two, three and four Kaluza-Klein and Landau levels: this will be done analytically for all modes. In the present case, they will allow us to compute the four-dimensional potential, find its minima and determine then the spectra and their symmetries. These technical results could be useful in more general scenarios than those considered here. For example, it has been suggested that unstable flux configurations can be associated with unstable (small angle) intersecting brane configurations [79]. In this context, our field theory approach can be seen as a classical approximation of a D-brane decay via open-string tachyon condensation [80].

Were $SU(N)$ the interesting gauge group, the field theory treatment just described would have been unnecessary, as pure theoretical arguments allow to argue the symmetries of the stable vacua. Indeed, on $\mathcal{T}^2$, the background with constant field strength at the origin of the Nielsen-Olesen instability, requires coordinate-dependent boundary conditions for fields. As we will prove and discuss in depth, for $SU(N)$ on a two torus, coordinate-dependent periodicity conditions are gauge equivalent to constant ones. In addition, all $SU(N)$ stable background on T^2 have zero field strength and then zero energy [81–83].

This result will allow us to show that all stable backgrounds are gauge equivalent to the trivial one at the classical level. The symmetries of the four-dimensional spectra, thus, can be directly inferred analyzing a system with constant periodicity conditions and trivial background.

The vacuum symmetries depend essentially on whether trivial or non-trivial ‘t Hooft fluxes are present. This translates then on whether the constant boundary conditions correspond to continous or discrete Wilson lines. While much literature is dedicated to the case of continous Wilson lines, one of the novel ingredients of this thesis is the phenomenological analysis of the pattern of gauge symmetry breaking and the spectrum of four-dimensional gauge and scalar excitations, for the general case of $SU(N)$ and discrete Wilson lines. The results will be shown to be consistent with those obtained from the field theory analysis of the effective Lagrangian, for the case of $SU(2)$, further supporting the consistency of the field theory approach developed in this work.

The last part of the work about flux compactification is dedicated to the analysis of the quantum stability of symmetry breaking. The fact that for $U(N)$ (or equivalently $SU(N)$ with non-trivial 't Hooft non-abelian flux) it is possible to interpret the stable vacuum configurations in terms of constant SS periodicity conditions, suggests that such symmetry breaking has a non-local nature. To understand and to clarify such point, we explicitly compute the one-loop effective potential using the Heat-Kernel technique. The Heat Kernel computation, because it takes place in coordinate space, results in a very useful instrument to distinguish contributions coming from local (ultra-violet sensitive) and non-local (ultra-violet insensitive) diagrams. The local contributions do not depend on the periodicity conditions and they are invariant under all the original symmetries. They do not contribute to the symmetry breaking order parameters. Only non-local contributions will be relevant for symmetry breaking, which is then protected from ultraviolet divergences.

Guideline

The thesis is organized as follows.

Chapter 1 and 2 provide a general introduction to field theory on compactified extra dimensions. We review the concept of compactification on 1- and 2-dimensional manifolds with and without fixed points. An important role is played by the discussion on chirality in the two different frameworks introduced before: orbifold (chapter 1) and flux (chapter 2) compactification. With respect to orbifold compactification, moreover, we recall the main symmetry breaking mechanisms as well as study the possible symmetry breaking patterns that can be achieved. Even if hopefully useful for a hypothetical novice reader, chapter 1 and 2 are rather technical; we therefore suggest to the reader interested in (phenomenological) applications to go directly to chapter 3.

Chapter 3,4,5 represent the core of the thesis and are dedicated to the study of gauge symmetry breaking in the context of flux compactification for both trivial and non-trivial 't Hooft non-abelian flux.

Chapter 3 and 4 are very complementary. In chapter 3, we look for the stable vacua and their symmetries for a $SU(N)$ gauge theory on a six-dimensional space-time where the two extra dimensions are compactified on a torus, using an effective field theory approach. In chapter 4, we adopt a more theoretical approach and prove how stable vacua and residual symmetries can be understood in terms of constant periodicity conditions. This result is valid also for the case of non-trivial 't Hooft non-abelian flux. In addition, we explicitly prove which symmetry breaking patterns can be achieved.

In chapter 5, instead, we analyze the stability of this symmetry breaking mechanism with respect to quantum corrections. In particular, we will show that its non-local nature makes it insensitive to energy scales greater than the compactification one. This chapter includes the computation of the one-loop potential using the Heat-Kernel technique, to take into account the effects stemming from non-local operators.

The last part of thesis is dedicated to the flavour problem. Chapter 6 contains a brief review of the SM flavour problem and of the Froggatt-Nielsen mechanism. Chapter 7 provides our implementation of a flavour model in the context of extra dimensions. Finally, in chapter 8 we conclude.

Chapter 1

Field theory on compactified flat extra dimensions

This chapter is divided in three parts. In the first, we will concentrate on the generalization to D dimensions of concepts such as space-time and the implementation of fermions and gauge theories. In the second part, we will introduce the concept of *compactification*. In particular, we will focus on:

1. Compactification on compact manifolds without singular points, with trivial and non-trivial boundary conditions. The latter is called *Scherk-Schwarz* compactification [58, 59].

2. Compactification on *Orbifolds* [45, 46], that is, on manifolds with singular points.

For illustration, we will provide some simple examples. The main result of this second part, is that orbifold compactification allows to obtain 4-dimensional effective theories containing chiral fermions.

In the third part of this chapter, we will consider symmetry breaking mechanisms in the context of orbifolding.

1.1 Generalities on flat extra dimensions

In this section, we recall a set of notions useful for the discussion of models with extra dimensions. In particular, we remind basical concepts such symmetries of flat space-time (subsection 1.1.1), fermions (subsection 1.1.2) and gauge theory (subsection 1.1.3) in D dimensions.

1.1.1 Space-time symmetries

We consider a D-dimensional homogeneous and isotropic space-time with $D = 4+d$. The vector $x^M \equiv (x^\mu, y^m)$ with $\mu = 0,1,2,3$ and $m = 1,...,d$, represents a generic point of this space.

In our discussion, we will concentrate only on *space-like* extra dimensions and we will work with the following metric:

$$\eta^{MN} = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & -1 & 0 & ... & 0 \\ 0 & 0 & -1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & -1 \end{pmatrix} . \tag{1.1}$$

The symmetries of the D-dimensional space-time with metric η^{MN} are all coordinate transformations $x^M \to x'^M$, under which the infinitesimal interval

$$ds^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 - y_1^2 - ... - y_d^2 = x^M \eta_{MN} x^N \tag{1.2}$$

is invariant. Since the space-time is homogeneous and isotropic, the only allowed transformations are the linear ones:

$$x^M \to x'^M = \mathcal{R}^{MN} x_N + a^M . \tag{1.3}$$

The matrices $\mathcal{R}$ have to satisfy

$$\mathcal{R}^T \eta \mathcal{R} = \eta \,, \qquad \text{or} \qquad \mathcal{R}^{-1} = \eta \mathcal{R}^T \eta. \tag{1.4}$$

The set of $D \times D$ matrices which satisfy the condition in eq.(1.4) constitutes the *Lorentz group in D dimensions*, $SO(1, D-1)$. One can verify that

- The product $\mathcal{R}_1\mathcal{R}_2$ of two elements satisfying the condition in eq.(1.4) is still a solution of the same condition:

$$(\mathcal{R}_1\mathcal{R}_2)^T \eta \mathcal{R}_1\mathcal{R}_2 = \mathcal{R}_2^T \mathcal{R}_1^T \eta \mathcal{R}_1 \mathcal{R}_2 = \eta . \tag{1.5}$$

- The identity is a solution of eq.(1.4):

$$\mathbb{I}^T \eta \mathbb{I} = \eta \text{ with } \mathbb{I}\eta = \eta \mathbb{I} . \tag{1.6}$$

- The inverse of any element, $\mathcal{R}^{-1}$, also satisfies the condition in eq.(1.4):

$$(\mathcal{R}^{-1})^T \eta \mathcal{R}^{-1} = \eta . \tag{1.7}$$

In addition, always from eq.(1.4), it is possible to deduce

$$\begin{aligned} \det(\mathcal{R}) &= \pm 1 \\ (\mathcal{R}^0_0)^2 - \sum_{i=1}^{D} (\mathcal{R}^i_0)^2 &= 1\,. \end{aligned} \tag{1.8}$$

The Lorentz group is, therefore, composed by four separated sectors characterized by $\det(\mathcal{R}) = \pm 1$ and $\mathcal{R}^0_0 = \pm 1$. The set of elements with $\det(\mathcal{R}) = 1$ and $\mathcal{R}^0_0 = 1$ constitutes the *proper Lorentz group*. The *D-dimensional Poincaré group* is constituted by transformations of the type in eq.(1.3) with $\mathcal{R}$ belonging to the Lorentz group.

1.1.2 Fermions

We briefly remind here how to construct spinorial representations of the Clifford algebra in the case of an arbitrary number of dimensions.[1].

In D dimensions, the Dirac matrices must satisfy the generalized Clifford algebra as follows

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}\,, \tag{1.9}$$

where η is defined in eq. (1.1) and $M, N = 0, 1, ..., D-1$.

Consider first the case of an even number of dimensions: $D = 2p + 2$. As first step, we construct $p+1$ fermionic operators of creation and destruction:

$$\left\{ \begin{array}{lcl} \Gamma^{\pm}_0 &=& \frac{1}{2}[\Gamma^0 \pm \Gamma^1] \\ \Gamma^{\pm}_\alpha &=& \frac{1}{2}[\Gamma^{2\alpha} \pm i\Gamma^{2\alpha+1}] \ \text{ with } \ \alpha = 1, ..., p\,, \end{array} \right. \tag{1.10}$$

which satisfy the anticommutation rules

$$\left\{ \begin{array}{lclcl} \{\Gamma^+_a, \Gamma^-_b\} &=& \delta^{ab} && \\ \{\Gamma^+_a, \Gamma^+_b\} &=& \{\Gamma^-_a, \Gamma^-_b\} &=& 0 \end{array} \right., \tag{1.11}$$

with $a, b = 0, 1, ..., p$. Eq.(1.11) is a direct consequence of the Clifford algebra in eq. (1.9). As second step, we define $p+1$ number operators

$$S^a = \Gamma^+_a \Gamma^-_a \qquad \text{with } \ a = 0, 1, .., p\,. \tag{1.12}$$

The eigenvalues of the number operator S^a are $0, 1$ as a consequence of the Clifford algebra.

[1]Throughout this work, we will use only of Dirac and Weyl fermions and then in this section we leave out the discussion about charge conjugation operator and Majorana fermions. A pedagogical introduction to these arguments can be found in [84–86].

The vacuum state is, therefore, defined in the following way:

$$\Gamma_a|\underbrace{0,0,...,0}_{p+1}>=0\ , \qquad \forall a\in[0,p]\ . \tag{1.13}$$

All possible states are thus obtained by applying the creation operators over the ground state:

$$|s_p,s_{p-1},...,s_0>=(\Gamma^+_p)^{s_p}(\Gamma^+_{p-1})^{s_{p-1}}...(\Gamma^+_0)^{s_0}\ |0,0,...0>\ , \tag{1.14}$$

with $s_a=0,1$. Unlike the bosonic oscillator case, the number of physical states is finite and equal to 2^{p+1}.

In this basis, the Γ^M matrices are $2^{p+1}\times 2^{p+1}$ matrices with matrix elements given by

$$<s'_p,s'_{p-1},...,s'_0|\Gamma^M|s_p,s_{p-1},...,s_0>\ . \tag{1.15}$$

Now we want to prove that the Dirac representation in eq.(1.14) (so called since it is built using the Dirac Γ matrices) is a spinorial representation of the D-dimensional Lorentz algebra.

In terms of the gamma matrices, the generators of the Lorentz group are defined in the following way,

$$\Sigma^{MN}=-\frac{i}{4}[\Gamma^M,\Gamma^N]\,. \tag{1.16}$$

For our purpose, it is sufficient to verify that the elements of Dirac basis, $|s_p,s_{p-1},...,s_0>$, are eigenstates of Σ^{MN} with seminteger eigenvalues. In terms of creation and destruction operators, $\Sigma^{2a,2a+1}$, for example, can be re-written as

$$\Sigma^{2a,2a+1}=(i)^{\delta_{a,0}}\Gamma^+_a\Gamma^-_a-\frac{1}{2}\ . \tag{1.17}$$

The generic state $|s_p,s_{p-1},...,s_0>$ is, therefore, an eigenstate of $\Sigma^{2a,2a+1}$ with eigenvalue $s_a-\frac{1}{2}=\pm\frac{1}{2}$.

In $D=2p+2$ dimensions the Dirac representation is a 2^{p+1}-dimensional spinor representation of the Lorentz algebra.

Moreover, in an *even* number of dimensions, the Dirac representation is a reducible representation of the Lorentz algebra. It is possible, indeed, to define an operator

$$\Gamma=i\Gamma^0\Gamma^1...\Gamma^{D-1} \tag{1.18}$$

which satisfies the following properties

$$\Gamma^2=\mathbf{1}\,, \tag{1.19}$$

$$\{\Gamma, \Gamma^M\} = 0 \; . \tag{1.20}$$

$$[\Gamma, \Sigma^{MN}] = 0 \; . \tag{1.21}$$

Eq.(1.19) implies that Γ eigenvalues (D-dimensional *chirality*) are necessarily ± 1. In addition, Γ and Σ^{MN} commute (eq. (1.21)) and, therefore, the generators of the Lorentz algebra Σ^{MN} *cannot connect two spinors of opposite D-dimensional chirality.*

The 2^p states of fixed chirality constitute a Weyl representation of the Lorentz algebra. The two Weyl representations (chirality$= \pm 1$) are not equivalent.

Consider, now, a theory with an odd number of dimensions $D' = D + 1 = 2p + 3$. In this case, the set of matrices which satisfy the new Clifford algebra is composed by the Γ^M matrices (with $M \leq D' - 2 = D - 1$) used in the $D = 2p + 2$ case, plus the new matrix

$$\Gamma^{D'-1} = i\Gamma \; , \tag{1.22}$$

with Γ defined in eq.(1.18).

In an *odd* number of dimensions, the Dirac representation is an irreducible representation of the Lorentz algebra. In fact, it is not possible in this case to define a chirality operator, that is a Γ' matrix which anticommutes with all the other Γ^M matrices:

$$\Gamma' = \underbrace{i\Gamma^0\Gamma^1 ... \Gamma^{D'-2}}_{\Gamma = -i\Gamma^{D'-1}} \Gamma^{D'-1} = -i\Gamma^{D'-1}\Gamma^{D'-1} = i\mathbf{1} \; . \tag{1.23}$$

Fermions in 5 dimensions

The $D = 5$ gamma matrices are given by the set of $D = 4$ gamma matrices γ^μ, with $\mu = 0, 1, 2, 3$, which satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \;\; \text{with} \;\; \eta^{\mu\nu} = Diag\,(1, -1, -1, -1) \; , \tag{1.24}$$

plus the matrix γ^4 defined as

$$\gamma^4 = i\gamma^5 = -\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 \, . \tag{1.25}$$

It is easy to prove that the matrices

$$\gamma^M = \left\{\gamma^\mu, \gamma^4\right\} \qquad \text{with } \mu = 0, 1, 2, 3 \qquad \text{and } M = 0, 1, 2, 3, 4 \, ,$$

satisfy the 5-dimensional Clifford algebra

$$\left\{\gamma^M, \gamma^N\right\} = 2\eta^{MN} \;\; \text{with} \;\; \eta^{MN} = Diag\,(1, -1, -1, -1, -1) \; .$$

From 4-dimensional point of view, a $D = 5$ Dirac representation of the Lorentz algebra is a vectorial representation, i.e. a spinor composed by the two different $D = 4$ chiralities.

Each $D=4$ chiral fermion carries two degrees of freedom and therefore a $D=5$ Dirac spinor has four degrees of freedom

$$\Psi^{5D} = \psi_L^{4D} + \psi_R^{4D} \,, \tag{1.26}$$

where ψ_L^{4D} and ψ_R^{4D} are the two $D=4$ different chiralities (left and right) defined

$$\begin{aligned} \psi_L^{4D} &= P_L \psi^{4D} = \frac{1-\gamma^5}{2}\psi^{4D} \\ \psi_R^{4D} &= P_R \psi^{4D} = \frac{1+\gamma^5}{2}\psi^{4D} \,. \end{aligned} \tag{1.27}$$

Fermions in 6 dimensions

The 6-dimensional Clifford algebra reads $\{\Gamma^M, \Gamma^N\} = \eta^{MN}$ with $\eta^{MN} = Diag\{1,-1,-1,-1,-1,-1\}$ and $M, N = 0, 1, .., 5$.

Our choice for the 6-dimensional Γ_M is the following

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes \mathcal{I} = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix} , \\ \Gamma^4 &= \gamma^5 \otimes i\sigma_1 = \begin{pmatrix} 0 & i\gamma^5 \\ i\gamma^5 & 0 \end{pmatrix} , \\ \Gamma^5 &= \gamma^5 \otimes i\sigma_2 = \begin{pmatrix} 0 & -\gamma^5 \\ \gamma^5 & 0 \end{pmatrix} , \end{aligned} \tag{1.28}$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and σ_1, σ_2 and σ_3 (the latter useful for the formulae below) are the Pauli matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1.29}$$

The 6-dimensional chirality operator Γ^7 is defined as

$$\Gamma^7 = -\Pi_{M=0}^5 \Gamma^M = \gamma^5 \otimes \sigma_3 = \begin{pmatrix} \gamma^5 & 0 \\ 0 & -\gamma^5 \end{pmatrix} , \tag{1.30}$$

and the projectors over 6-dimensional fixed-chirality states are

$$\begin{aligned} \mathcal{P}_L &= \frac{1-\Gamma^7}{2} = \begin{pmatrix} P_L & 0 \\ 0 & P_R \end{pmatrix} \\ \mathcal{P}_R &= \frac{1+\Gamma^7}{2} = \begin{pmatrix} P_R & 0 \\ 0 & P_L \end{pmatrix} . \end{aligned} \tag{1.31}$$

P_L and P_R are the projectors over 4-dimensional fixed-chirality states defined in eq.(1.27).

A $D = 6$ Dirac spinor is an 8-dimensional vector that can be parametrized in terms of two $D = 4$ Dirac fermions ψ and χ:

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix} . \tag{1.32}$$

As in $D = 4$ case, we can define a $D = 6$ Dirac fermion as direct sum of two Weyl fermions:

$$\Psi_D = (\mathcal{P}_L + \mathcal{P}_R)\Psi = \Psi_L + \Psi_R \ , \tag{1.33}$$

where

$$\begin{aligned} \Psi_L &= \mathcal{P}_L \Psi = \begin{pmatrix} P_L & 0 \\ 0 & P_R \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = P_L\psi \oplus P_R\chi = \psi_L \oplus \chi_R \\ \Psi_R &= \mathcal{P}_R \Psi = \begin{pmatrix} P_R & 0 \\ 0 & P_L \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = P_R\psi \oplus P_L\chi = \psi_R \oplus \chi_L \ . \end{aligned} \tag{1.34}$$

A $D = 6$ Weyl fermion is, therefore, built up with 2 $D = 4$ Weyl spinors with opposite 4-dimensional chirality.

1.1.3 $SU(N)$ gauge theory

Let's concentrate on the group $SU(N)$. The vector potential $A_M(x, y)$ with $M = 0, 1, ..., D - 1$ is a field in the adjoint representation of the gauge group and can be parametrized

$$A_M = A^a_M T^a \ , \tag{1.35}$$

where T^a, with $a = 1, 2, .., N^2 - 1$, are the $SU(N)$ generators which satisfy the following conditions:

1. $T^\dagger_a = T_a$.
2. Tr(T_a)= 0.
3. Tr$(T^a T^b) = \frac{1}{2}\delta^{ab}$.

The extra components of the vector potential, A_i with $i = 4, ..., D-1$, are called *internal components*.

The generic gauge transformation takes the form

$$U(x, y) \equiv e^{i\alpha^a(x,y)T^a} \ , \tag{1.36}$$

where the gauge parameters $\alpha^a(x,y)$ are scalar fields with respect to the D dimensional Lorentz group.

The behaviour of the vector potential A_M under gauge transformation reads

$$A_M \to A'_M \equiv U A_M U^+ - \frac{i}{g} U \partial_M U^+ \ . \tag{1.37}$$

The action which describes the kinetic terms of A_M and their selfinteractions is given by

$$S \equiv -\frac{1}{2} \int d^D x \ \mathrm{Tr}\left[F^{MN} F_{MN}\right] \ , \tag{1.38}$$

where

$$F_{MN} = \partial_M A_N - \partial_N A_M - ig[A_M, A_N] \ . \tag{1.39}$$

Under gauge transformations, it results

$$F_{MN} \to U F_{MN} U^+ \tag{1.40}$$

and therefore it is straightforward to prove that the action in eq.(1.37) is gauge invariant.

The gauge action in eq.(1.38) sets the dimensions of fields: A_M carries dimension in energy equal to $(1 + \frac{d}{2})$ where d is the number of the extra dimensions. Therefore, the *gauge constant carries* ***negative*** *dimension in energy equal to* $-\frac{d}{2}$.

Now, let's consider fermionic matter fields ψ belonging to the representation r of $SU(N)$. Under gauge transformations, the fermionic fields transform in the following way:

$$\psi(x,y) \to \psi'(x,y) \equiv e^{i\alpha^a(x,y) T_r^a} \psi(x,y) \ , \tag{1.41}$$

where T_r^a are the $SU(N)$ generators in the representation r. The action which describes the kinetic terms of fermionic fields and their interactions with gauge fields is given by

$$S = \int d^D x \ i \ \overline{\psi} \ \Gamma^M \ D_M \ \psi \ , \tag{1.42}$$

where D_M is the covariant derivative defined

$$D_M \equiv \partial_M - ig A_M \ , \tag{1.43}$$

with $A_M = A_M^a T_r^a$. From eq.(1.42) and eq.(1.43), it is possible to verify that fermionic fields carry dimension in energy equal to $\frac{D-1}{2}$.

1.2 Compactification

Let's consider a theory with a D-dimensional space-time, where $D = 4 + d$. We denote with $\mathcal{M}_4$ the ordinary Minkowski four-dimensional space-time and with C the d-dimensional manifold on which the extra dimensions y^m are compactified.

In general, a compact space C can be represented as $C = M/K$ where M is a non compact manifold and K is a discrete group of symmetry that preserves the metric of C. K acts on M through the operators τ_k: $M \to M$ for $k \in K$:

$$K \; : \; y \to \tau_k[y] \; . \tag{1.44}$$

τ_k is the representation of the group K in the coordinate space, which means that $\tau_{k1} \cdot \tau_{k2} = \tau_{k1\,k2}$.

To *compactify* the d extra dimensions means to identify the points of M connected by transformations belonging to the group K:

$$y \equiv \tau_k[y] \; . \tag{1.45}$$

The point of M which are invariant (fixed) under the action of K are called *singular points* of the manifold C.

Now we want to investigate which discrete symmetries (K) can be used to compactify d flat extra dimensions.

The element $ds^2 = \eta^{MN} \; dx_M \; dx_N$ has to be invariant with respect to the action of this discrete symmetry group. As we have discussed in section 1.1.1, in the case of homogeneous and isotropic space-time, the symmetry group of ds^2 coincides with the group of linear transformations of coordinates. Concentrating on transformations which leave invariant the ordinary 4 dimensions, we have

$$\begin{aligned} x^\mu &\to x^\mu \\ y^m &\to \mathcal{R}^{mn} y^n + a^m \; , \end{aligned} \tag{1.46}$$

with $\mathcal{R}^{mn}$ satisfying[2]

$$\mathcal{R}^T \mathcal{R} \;\; = \;\; \mathbf{1} \; . \tag{1.47}$$

Therefore, the discrete symmetry group K has to be a discrete subgroup of the d-dimensional rotations, $SO(d)$, and the d-dimensional translations.

The simplest case is when such symmetry group coincides with the group of the translations by vectors of a fixed d dimensional lattice Λ. The compact manifold that we obtain

[2]Eq. (1.47) is the conditions that appears in eq.(1.4) reduced to the extra dimensions.

in this case is the d-dimensional torus $\mathcal{T}^d$ defined by[3]

$$\mathcal{T}^d = \frac{\mathbb{R}^d}{\Lambda} \,. \tag{1.48}$$

Hence, in $\mathcal{T}^d$ the points y and $y+V$, with $V \in \Lambda$, are identified. Since the group of translations by lattice vectors acts freely, the *torus has no singular points.*

More complicated manifolds are obtained compactifying the extra dimensions using the total symmetry group $K = \Lambda \cup G$, that is, using at the same time translations by lattice vectors and a discrete subgroup G of the d-dimensional rotations $SO(d)$. The only subgroups of $SO(d)$ that can be used, are the discrete subgroups that act crystallographically on the torus lattice Λ. To act crystallographically on the torus lattice means that for $V \in \Lambda$ and $g \in G$, gV still belongs to the lattice Λ. Such subgroups of $SO(d)$, in general, coincide with $\mathbb{Z}_N$ groups for some fixed values of N. The resulting compact space obtained in this way is denoted by

$$C = \frac{\mathcal{T}^d}{\mathbb{Z}_N} \,. \tag{1.49}$$

There exist always points that are fixed (invariant) under rotations belonging to $\mathbb{Z}_N$. The resulting compact manifold, therefore, has singular points and it is called *orbifold*. We denote with θ_n, with $n = 0, ..., N-1$, the elements of $\mathbb{Z}_N$ which satisfy the group conditions $(\theta_n)^N = 1$ and use τ_{θ_n} for its representation in the coordinate space. The set of points $\{\vec{y} = p_{i_n}\}$ which is left fixed by an element θ_n of the orbifold group $\mathbb{Z}_N$ depends on n. Since θ_{N-n} is the inverse of θ_n, the fixed points in the sectors n and $N-n$ are the same, and their number is an invariant quantity defined

$$\mathcal{N}_{p_{i_n}} = \det(1-\theta) = \prod_{j=1}^{d/2} 4\sin^2 \pi \frac{n_j}{N} \,. \tag{1.50}$$

Moreover, the sector $n = 0$ is trivial, and has of course no fixed points. The physically distinct and relevant sectors are, therefore, labelled by $n = 0, 1, ..., [N/2]$, where [..] denotes the integer part.

We emphasize that a generic fixed point p_{i_n} is left fixed by the element θ_n only modulo a suitable translation. More precisely, bringing back the image $\tau_{\theta_n}[p_{i_n}]$ to the original p_{i_n} will require some integer numbers $q_{i_k\,a}$ of translations along the basis vectors e_a with $a = 1, ..., d$ of the torus lattice Λ, so that

$$p_{i_n} = \tau_{\theta_n}[p_{i_n}] + \sum_a q_{i_n\,a} e_a \,. \tag{1.51}$$

[3] In the case $d = 1$, the compact space coincides with a circle and we will use the standard notation S_1.

The numbers $q_{i_n\,a}$ depend on the particular fixed point p_{i_n} and are, in general, different for the different fixed points of the same element θ_n. This result gives us a set of constraints among orbifold projections and translations: at a given θ_n fixed point, p_{i_n}, with associated integers $q_{i_n\,a}$, the effective orbifold projection is implemented not just by θ_n but rather by

$$\theta_{i_n} = \prod_a (T_a)^{q_{i_n\,a}}\, \theta_n \,, \tag{1.52}$$

where T_a is the fundamental translation along e_a. θ_n itself can be interpreted as the effective orbifold projection at the origin when $q_{i_n\,a} = 0\ \forall a$.

The consistency conditions in eq.(1.52) have important consequences that will play a fundamental role in the discussion of symmetry breaking (see section 1.3). In fact, also θ_{i_n} defined in eq.(1.52) is an element of the orbifold group $\mathbb{Z}_N$ and so has to satisfy

$$(\theta_{i_n})^N = \left(\prod_a (T_a)^{q_{i_n\,a}}\, \theta_n\right)^N = 1\,. \tag{1.53}$$

To eq.(1.53), we have to add

1. The consistency condition among translations
$$[T_a, T_b] = 0\,, \qquad\qquad \forall a, b = 1, ..., d\ , \tag{1.54}$$
where d is the number of the extra dimensions.

2. The consistency condition depending on how the basis vectors e_a of the torus lattice are mapped within each other by the orbifold rotation.

Now we want to particularize the general results of this section to the case of five and six dimensions: $S_1/\mathbb{Z}_2$ and $\mathcal{T}^2/\mathbb{Z}_N$.

$S_1/\mathbb{Z}_2$

In the case $d = 1$ (5 dimensions), the generic element of the complete discrete group $K = \Lambda \cup G$ which leaves invariant the ordinary 4 dimensions and preserves the metric is simply

$$y \to \mathcal{R}y + d\ , \tag{1.55}$$

with $\mathcal{R}$ satisfying the condition of eq.(1.47), $\mathcal{R}^2 = 1$. $\mathcal{R}$ has to be an element of the discrete group $G = \mathbb{Z}_2$.

Therefore, the only non trivial symmetries that we can impose on the extra dimension are:

$$y \xrightarrow{T} y + 2\pi R \tag{1.56}$$
$$y \xrightarrow{\theta} -y\ , \tag{1.57}$$

where R is the compactification radius. Modding out the real line by the translations, eq.(1.56), we obtain the circle S_1. The orbifold projection is defined by identifying points of S_1 that are related by a $\mathbb{Z}_2$ reflection, eq.(1.57). The resulting compact space is called S_1/Z_2. There are two fixed points $p_0 = 0$ and $p_1 = \pi R$ and the physical part of the internal space is the segment of length πR that connects them. The fixed points $p_0 = 0$ and $p_1 = \pi R$ have $q_0 = 0$ and $q_1 = 1$ in eq.(1.51) and the corresponding effective projections are

$$\begin{aligned} \theta_0 &= \theta \\ \theta_1 &= T\,\theta\,, \end{aligned} \tag{1.58}$$

where T is the translation of $2\pi R$ along the fifth dimension. θ_1 is an element of $\mathbb{Z}_2$ and then has to verify $(\theta_1)^2 = 1$. In this way we obtain the consistency condition of $S_1/\mathbb{Z}_2$

$$(T\,\theta)^2 = 1\,. \tag{1.59}$$

All the other consistency conditions discussed in the introduction are trivial in the case of only one extra dimension.

To summarize, the compact space $S_1/\mathbb{Z}_2$ is invariant under three geometric symmetries: the translations T and the reflections θ_0 and θ_1 at the fixed points $p_0 = 0$ and $p_1 = \pi R$, respectively. Eq.(1.58) implies that these symmetries are not independent and that we can concentrate only on two of them. For example, T and θ_0 with the consistency condition in eq.(1.59).

$\mathcal{T}^2/\mathbb{Z}_N$

A torus $\mathcal{T}^2$ is described by three real parameters, for instance two compactification radii and one angle $\{R_1, R_2, \alpha\}$, and is obtained by identifying points which are related by the two translations $T_a : \vec{y} \to \vec{y} + e_a$ along the basis vector $e_1 = 2\pi R_1$ and $e_2 = 2\pi R_1 U$ with $U = R_2/R_1 e^{i\alpha}$. The angle α is the angle between e_1 and e_2. The orbifold $\mathcal{T}^2/\mathbb{Z}_N$ is obtained by further identifying points of the torus related by the rotations $\theta_n \in \mathbb{Z}_N$ of an angle $\frac{2\pi n}{N}$:

$$\theta_n : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} -> \begin{pmatrix} \cos\frac{2\pi n}{N} & \sin\frac{2\pi n}{N} \\ -\sin\frac{2\pi n}{N} & \cos\frac{2\pi n}{N} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\,. \tag{1.60}$$

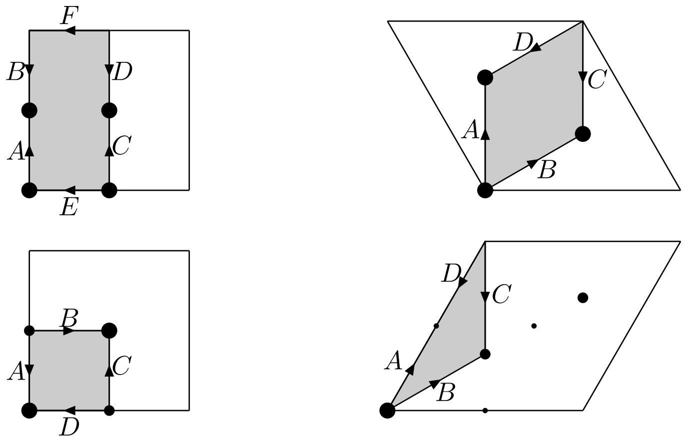

Figure 1.1: The picture shows the $\mathcal{T}^2/\mathbb{Z}_2$ $\mathcal{T}^2/\mathbb{Z}_3$, $\mathcal{T}^2/\mathbb{Z}_4$ and $\mathcal{T}^2/\mathbb{Z}_6$ orbifolds and their covering tori. Points of decreasing size indicate the θ_1, θ_2 and θ_3 fixed points respectively. The grey region represents the fundamental domain of the orbifolds, and the segments delimiting it must be identified according to: $A \sim D$, $B \sim C$ and, in the $\mathcal{T}^2/\mathbb{Z}_2$ case, $E \sim F$.

We want to deduce which values of N can be used to compactify. We discuss the general case of an even number d of extra dimensions and then particularize the result to the $d = 2$ case. The discrete group $\mathbb{Z}_N$ is the subgroup of $SO(d)$ composed by the rotations $\theta \in \mathbb{Z}_N$ such that $\theta^N = 1$. θ has eigenvalues $e^{\pm 2\pi i \frac{n_i}{N}}$ where $n_i = 0, 1, ..., N-1$ and $i = 1, .., d/2$.

Imposing that $\mathbb{Z}_N$ acts crystallographically on the torus lattice, the possible values of N are reduced. In fact, this implies that the quantity

$$\mathrm{Tr}\theta = \sum_{i=1}^{d/2} 2\cos\frac{2\pi n_i}{N} \tag{1.61}$$

must be an integer, see eq.(1.60) . The requirement of crystallographic action is very restrictive. For example, for $d = 2$, only $N = 2, 3, 4, 6$ are allowed. Therefore the only possible 6-dimensional orbifold compactifications are $\mathcal{T}^2/\mathbb{Z}_2$, $\mathcal{T}^2/\mathbb{Z}_3$, $\mathcal{T}^2/\mathbb{Z}_4$ and $\mathcal{T}^2/\mathbb{Z}_6$, see fig.(1.1)

The case $N = 2$ is consistent for arbitrary values of the three real parameters of torus R_1, R_2 and α and corresponds to a rather straightforward generalization of the 1-dimensional case $S_1/\mathbb{Z}_2$. The cases $N = 3, 4, 6$ are instead consistent only when $R_1 = R_2$ and $\alpha = \frac{2\pi n}{N}$ with $n = 1, .., N-1$. The fundamental domain of these $\mathcal{T}^2/\mathbb{Z}_N$ orbifolds can be chosen to be a polygon of surface $|e_1 \wedge e_2|/N$ connecting the different fixed points.

We want, now, to particularize the condition in eq.(1.53) to the $\mathcal{T}^2/\mathbb{Z}_N$ case. We denote with $\theta = \theta_{n=1}$, the smallest non trivial orbifold rotation. In this way, all the other elements of $\mathbb{Z}_N$ can be written as

$$\theta_n = (\theta)^n\ . \tag{1.62}$$

In this notation and in the $\mathcal{T}^2/\mathbb{Z}_N$ case, eq.(1.53) reads:

$$\left(T_1^{q_{1i}} T_2^{q_{2i}} \theta^n\right)^{N/n} = 1 \tag{1.63}$$

for each integer $n = 1, .., [N/2]$ such that $\mathbb{Z}_{N/n}$ is a subgroup of $\mathbb{Z}_N$. The index i distinguishes the different fixed points of a given sector n and q_{1_i} and q_{2_i} are different integer numbers for different fixed points in the same θ_n sector.

The consistency condition in eq.(1.54), now, reads

$$[T_1, T_2] = 0\,. \tag{1.64}$$

There is an additional condition depending on how the basis vectors e_a are mapped within each other by the orbifold rotation. For $N = 2$, each e_a is reflected to $-e_a$, and thus $\theta T_a = (T_a)^{-1}\theta$, but this does not lead to any new condition. For $N = 3, 4, 6$, the torus angle is $\alpha = \frac{2\pi}{N}$ and then one has $\theta e_1 = e_2$, and, hence

$$\theta T_1 = T_2 \theta\,. \tag{1.65}$$

Notice that, as in the case of $S_1/\mathbb{Z}_2$, it is possible to use the relations in eqs.(1.63)-(1.64)-(1.65) in order to reduce the number of independent geometric symmetries of the orbifolds.

To make more clear this statement, we analyze explicitly the $\mathcal{T}^2/\mathbb{Z}_3$ case with $\theta = 2\pi/3$. In this case, there are three fixed points $p_0 = (0, 0)$, $p_1 = (\pi, \pi/\sqrt{3})$ and $p_2 = (0, 2\pi/\sqrt{3})$ (For simplicity, here we have fixed $R_1 = R_2 = 1$). The lattice is, a priori, compatible with five different symmetries: two translations (T_1, T_2) and the three orbifold rotations of an angle $2\pi/3$ at the three fixed points $(\theta_0, \theta_1, \theta_2)$. In this case the geometric consistency conditions in eq.(1.63) take the following form:

$$\begin{aligned}
\theta_1 &= T_1\theta_0 \\
\theta_2 &= T_1 T_2 \theta_0 \\
T_1\theta_0 &= \theta_0 T_2 \\
T_1 T_2 &= T_2 T_1\;.
\end{aligned} \tag{1.66}$$

Therefore, we can write all symmetries in terms of either two orbifolds or an orbifold and a translation. For instance, we can use T_1 and θ_0 as follows

$$\begin{aligned}
\theta_1 &= T_1\theta_0 \\
\theta_2 &= T_1\theta_0 T_1^{-1} \\
T_2 &= \theta_0 T_1 \theta_0^{-1}\;.
\end{aligned} \tag{1.67}$$

Notice that T_1 and θ_0 are not independent since they are required to satisfy

$$\begin{aligned}
(\theta_1)^3 &= (T_1\theta_0)^3 = 1 \\
(\theta_2)^3 &= (T_1\theta_0 T_1^{-1})^3 = 1\,.
\end{aligned} \tag{1.68}$$

This result will be important in the discussion about the gauge symmetry breaking.

The result obtained for $\mathcal{T}^2/\mathbb{Z}_3$ is valid for all $\mathcal{T}^2/\mathbb{Z}_N$ with $N = 3, 4, 6$. In all cases, we begin with a lattice which is invariant under two translations (T_1, T_2) and under the set of orbifold rotations at the fixed points. To describe the properties of the geometric symmetry of the lattice, we can always restrict to use, for instance, only one translation and one orbifold rotation with constraints of the type of eq.(1.68). This result reflects the fact that, unlike $\mathcal{T}^2/\mathbb{Z}_2$ case, in $\mathcal{T}^2/\mathbb{Z}_N$ with $N = 3, 4, 6$ the lattice has only one independent radius.

1.2.1 Field theory on manifold without fixed points

Here, we compactify the d extra dimensions using a discrete symmetry group K acting freely on the non-compact manifold $M = \mathbb{R}^d$. In particular, we choose K as the group of translations of a fixed torus lattice. The compact space C is built identifying the points of M connected by the transformations belonging to K, eq. (1.45). The physics will depend only on the points of C: if $\mathcal{L}_D\ [\psi(x, y)]$ is the Lagrangian which describes the D-dimensional theory, it must satisfy the following property

$$\mathcal{L}_D\ [\psi(x, y)] = \mathcal{L}_D\ [\psi(x, \tau_k[y])] + \partial_M f(\psi(x, \tau_k[y])\ , \tag{1.69}$$

where $\partial_M f(\psi(x, \tau_k[y])$ does not affect the total action:

$$\int d^D x \partial_M f(\psi(x, \tau_k[y]) = 0\,. \tag{1.70}$$

The necessary condition to satisfy the condition in eq.(1.69) is

$$\psi(x, y) = T_k \psi(x, \tau_k[y])\ , \tag{1.71}$$

where T_k is an element belonging to the (global or local) symmetry group of the action. This condition is known as *Scherk-Schwarz* (SS) *compactification.* The particular case in which $T_k = 1$ (trivial boundary conditions) is known as *ordinary compactifications.* Notice that the SS boundary conditions are motivated by the fact that two field configurations $T_k\psi(x, \tau_k[y])$ and $\psi(x, y)$, related by a symmetry transformation of the action, are equivalent.

In the following examples, we concentrate on the ordinary compactification: we illustrate the standard technique to obtain a 4-dimensional Lagrangian coming from a D-dimensional one and prove that compactifying on a manifold without singular points and without gauge (or gravitational) background, it is not possible to obtain $D = 4$ chiral fermions.

The study of the Scherk-Schwarz compactification is postponed to the next section in which we will analyze symmetry breaking mechanisms in the context of extra dimensions.

Example: Free fermion on a torus, a (4D) vectorial theory

Consider a free fermion living in a 6 dimensional space-time in which the two extra dimensions are compactified on an orthogonal torus of radii R_1 and R_2. In this case, the manifold M coincides with $\mathbb{R}^2$ whereas the discrete group of transformations K with the translations by lattice vectors of length $2\pi R_1$ and $2\pi R_2$.

The representation in the coordinate space of the k-th element of the group K is given by

$$\tau_k\,[\vec{y}] = \begin{pmatrix} y_1 + 2\pi k R_1 \\ y_2 + 2\pi k R_2 \end{pmatrix} \quad \text{with} \quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \,. \tag{1.72}$$

In order to compactify, we identify the points of M connected by a discrete transformation τ_k:

$$\begin{aligned} y_1 &\equiv \vec{y} + 2\pi k R_1 \\ y_2 &\equiv \vec{y} + 2\pi k R_2 \,. \end{aligned} \tag{1.73}$$

The action of the discrete symmetry group K on fermionic fields is trivial and, therefore, denoting by T_k the representation in the Dirac space of K, it results $T_k = 1, \forall\, k \in\, K$.

The boundary conditions, in this case, reduce to simple periodicity conditions of the type

$$\begin{aligned} \Psi(x, y_1 + 2\pi R_1, y_2) &= \Psi(x, y_1, y_2) \\ \Psi(x, y_1, y_2 + 2\pi R_2) &= \Psi(x, y_1, y_2) \,, \end{aligned} \tag{1.74}$$

where Ψ is a $D = 6$ Dirac fermion that can be parametrized in terms of $D = 4$ Dirac fermions as in eq.(1.32).

Fields satisfying boundary conditions of the type in eq.(1.74), admit the following Fourier expansions

$$\Psi = \begin{pmatrix} \psi(x, y_1, y_2) \\ \chi(x, y_1, y_2) \end{pmatrix} = \frac{1}{2\pi\sqrt{R_1 R_2}} \sum_{n,m=-\infty}^{\infty} \begin{pmatrix} \psi^{(n,m)}(x) \\ \chi^{(n,m)}(x) \end{pmatrix} e^{i\frac{ny}{R_1}}\, e^{i\frac{my}{R_2}} \,. \tag{1.75}$$

The coefficients $\psi^{(n,m)}(x)$ and $\chi^{(n,m)}(x)$ depend only on the ordinary four dimensions and are called Kaluza-Klein modes (KK modes).

The factor $1/(2\pi\sqrt{R_1 R_2})$ is a wave function normalization factor with respect to the integral over the torus. The $D = 6$ fermion carries dimensions in energy equal to 5/2 and the $D = 4$ fields $\psi^{(n,m)}(x)$ and $\chi^{(n,m)}(x)$ carry dimensions in energy equal to 3/2.

Consider a 6-dimensional free massless fermion described by the following Lagrangian

$$\mathcal{L}^{6D} = i\overline{\Psi}\Gamma^M \partial_M \Psi \,, \tag{1.76}$$

where the $D = 6$ Γ matrices are given in eq.(1.28). Replacing each field with its Fourier expansion and integrating over the torus surface, we obtain the following 4-dimensional effective Lagrangian:

$$\begin{aligned}
\mathcal{L}^{4D} &= \sum_{n,m=-\infty}^{\infty} \Big[i\overline{\psi}^{\,(-n,-m)}(x)\gamma^\mu\partial_\mu\psi^{\,(n,m)}(x) + i\overline{\chi}^{\,(-n,-m)}(x)\gamma^\mu\partial_\mu\chi^{\,(n,m)}(x) \\
&- \overline{\psi}^{\,(-n,-m)}(x)i\gamma^5\left(\frac{n}{R_1}+i\frac{m}{R_2}\right)\chi^{\,(n,m)}(x) \\
&- \overline{\chi}^{\,(-n,-m)}(x)i\gamma^5\left(\frac{n}{R_1}-i\frac{m}{R_2}\right)\psi^{\,(n,m)}(x)\Big] \ .
\end{aligned} \tag{1.77}$$

For each fixed pair of n and m, the effective 4-dimensional mass is given by

$$M = i\gamma^5 \begin{pmatrix} 0 & \left(\frac{n}{R_1}+i\frac{m}{R_2}\right) \\ \left(\frac{n}{R_1}-i\frac{m}{R_2}\right) & 0 \end{pmatrix} . \tag{1.78}$$

The physical square mass $MM^\dagger$ has eigenvalues $\left(\frac{n}{R_1}\right)^2 + \left(\frac{m}{R_2}\right)^2$. Therefore, beginning with a 6-dimensional massless Dirac fermion and compactifying trivially the two extra dimensions on a torus, we have obtained a 4 dimensional effective theory in which we can observe a *tower* of KK modes with square masses $\left(\frac{n}{R_1}\right)^2 + \left(\frac{m}{R_2}\right)^2$. Only the zero mode $(n = m = 0)$ remains massless.

Massless modes have constant extra-dimensional wave function. They can arise only from fields satisfying trivial boundary conditions. For compactification on a torus[4], all $D = 4$ Dirac fermions (regardless of their 4-dimensional chirality) contained in the original $D = 6$ Dirac fermion, satisfy trivial boundary conditions and then admit zero modes at the same time, see eq.(1.77). In this case it is not possible to obtain $D = 4$ chiral fermions or in other words, it is not possible to have zero modes with only a fixed chirality.

Example: $U(1)$ gauge theory on a torus T^2 and unitary gauge

We discuss, here, only the abelian case. The result (including the discussion about the unitary gauge) can be, however, extended to the non-abelian case. The 6-dimensional

[4] In the next chapter we will show that the situation is very different in presence of a non-trivial gauge background.

$U(1)$ Lagrangian reads[5]:

$$\begin{aligned}\mathcal{L}^{6D} &= -\frac{1}{2}\Bigg[\sum_{\mu>\nu} F_{\mu\nu}F^{\mu\nu} + (\partial_\mu A_5)(\partial^\mu A^5) + (\partial_\mu A_6)(\partial^\mu A^6) \\ &+ (\partial_5 A_\mu)(\partial^5 A^\mu) + (\partial_6 A_\mu)(\partial^6 A^\mu) + (\partial_5 A_6 - \partial_6 A_5)(\partial^5 A^6 - \partial^6 A^5) \\ &\quad -2A_\mu \partial^\mu(\partial_5 A^5 + \partial_6 A^6)\Bigg] \ . \end{aligned} \tag{1.79}$$

The two extra dimensions are compactified on an orthogonal torus. The gauge fields A_M satisfying trivial periodicity conditions,

$$\begin{aligned} A_M(x, y_1 + 2\pi R_1, y_2) &= A_M(x, y_1, y_2) \ , \\ A_M(x, y_1, y_2 + 2\pi R_2) &= A_M(x, y_1, y_2) \ , \end{aligned} \tag{1.80}$$

can be expanded in Fourier series in the following way:

$$A_M(x,y) = \frac{1}{2\pi\sqrt{R_1 R_2}} \sum_{n,m=-\infty}^{\infty} A_M^{(n,m)}(x) e^{i\frac{ny_1}{R_1}} e^{i\frac{my_2}{R_2}} \ . \tag{1.81}$$

Also in this case, the coefficients $A_M^{(n,m)}$ depend only on the ordinary 4 dimensions and constitute the KK modes of $A_M(x,y)$. Replacing in eq.(1.79), each field with its KK expansion and integrating over the extra dimensions, one obtains the following 4-dimensional Lagrangian:

$$\begin{aligned}\mathcal{L}_{4D} &= -\frac{1}{4} F^{(0,0)}_{\mu\nu}(x) F^{\mu\nu}_{(0,0)}(x) \\ &- \frac{1}{2} \sum_{\bar{n},\bar{m}=-\infty}^{\infty} \Bigg[\sum_{\mu>\nu} F^{(-\bar{n},-\bar{m})}_{\mu\nu}(x) F^{\mu\nu}_{(\bar{n},\bar{m})}(x) - M^2_{(\bar{n},\bar{m})} A^{(-\bar{n},-\bar{m})}_\mu(x) A^\mu_{(\bar{n},\bar{m})}(x) \\ &- \left(\partial_\mu A^{(-\bar{n},-\bar{m})}(x)\right)\left((\partial^\mu A^{(\bar{n},\bar{m})}(x)\right) + M^2_{(\bar{n},\bar{m})} A^{(-\bar{n},-\bar{m})}(x) A^{(\bar{n},\bar{m})}(x) \\ &- \left(\partial_\mu a^{(-\bar{n},-\bar{m})}(x)\right)\left(\partial^\mu a^{(\bar{n},\bar{m})}(x)\right) - 2iM_{(\bar{n},\bar{m})} A^{(-\bar{n},-\bar{m})}_\mu(x)\left(\partial^\mu a^{(\bar{n},\bar{m})}(x)\right)\Bigg] \\ &- \frac{1}{2}\left(\partial_\mu A_5^{(0,0)}(x)\right)\left(\partial^\mu A^5_{(0,0)}(x)\right) - \frac{1}{2}\left(\partial_\mu A_6^{(0,0)}(x)\right)\left(\partial^\mu A^6_{(0,0)}(x)\right) \ , \end{aligned} \tag{1.82}$$

where $\sum_{\bar{n},\bar{m}=-\infty}^{\infty}$ denotes the sum over all n and m except the case $n = m = 0$ and

$$M_{(n,m)} = \sqrt{\frac{n^2}{R_1^2} + \frac{m^2}{R_2^2}} \ , \tag{1.83}$$

[5]We denote now by ∂_5, ∂_6 the derivatives with respect to the fifth and sixth dimension and by A_5, A_6 the component of the vectorial field A_M along the fifth and sixth dimension.

$$\cos\theta_{(n,m)} = \frac{\frac{n}{R_1}}{\sqrt{\frac{n^2}{R_1^2}+\frac{m^2}{R_2^2}}}\ , \qquad \sin\theta_{(n,m)} = \frac{\frac{m}{R_2}}{\sqrt{\frac{n^2}{R_1^2}+\frac{m^2}{R_2^2}}}\,. \tag{1.84}$$

The 4-dimensional fields $a^{(n,m)}$ and $A^{(n,m)}$ are defined as

$$\begin{aligned} a^{(n,m)}(x) &= \cos\theta_{(n,m)}\, A_5^{(n,m)}(x) + \sin\theta_{(n,m)} A_6^{(n,m)}(x)\,, \\ A^{(n,m)}(x) &= -\sin\theta_{(n,m)}\, A_5^{(n,m)}(x) + \cos\theta_{(n,m)} A_6^{(n,m)}(x)\,. \end{aligned}$$

The Lagrangian in eq.(1.82) describes a theory with a tower of massive gauge bosons $A_\mu^{(n,m)}$ interacting by derivative couplings with the massless scalar $a^{(n,m)}$. The fact that the fields $a^{(n,m)}$ are massless and interact only by derivative couplings with the corresponding massive gauge bosons $A_\mu^{(n,m)}$, suggests that the fields $a^{(n,m)}$ play the role of pseudo-Goldstone bosons in the breaking of the $U(1)$ symmetry related to the $A_\mu^{(n,m)}$.

Now we want to show that it is possible to fix the gauge related to $A_\mu^{(n,m)}$ in such a way that the pseudo Goldstone bosons do not appear in the resulting Lagrangian. This gauge is called *unitary gauge.*

The 6-dimensional vector potential have the following properties under six-dimensional $U(1)$ gauge transformations

$$A_M(x,y_1,y_2) \ \rightarrow\ A'_M(x,y_1,y_2) = A_M(x,y_1,y_2) + \partial_M \alpha(x,y_1,y_2)\,. \tag{1.85}$$

The gauge transformed A'_μ has to be invariant under translations of $2\pi R_1$ and $2\pi R_2$ along the fifth and sixth dimension respectively; this implies that the gauge parameter α has to be periodic too and can be expanded in Fourier series as follows

$$\alpha(x,y_1,y_2) = \sum_{n,m=-\infty}^{\infty} \alpha^{(n,m)}(x) e^{2\pi i \frac{n}{R_1} y_1} e^{2\pi i \frac{m}{R_2} y_2}\,. \tag{1.86}$$

Replacing each field of eq.(1.85) with its Fourier expansion, it is possible to deduce the gauge transformation rules of the 4-dimensional fields $A_\mu^{(n,m)}$, $A_5^{(n,m)}$ and $A_6^{(n,m)}$

$$\begin{aligned} A_\mu^{(n,m)}(x) &\rightarrow A_\mu^{(n,m)}(x) + \partial_\mu \alpha^{(n,m)}(x)\,, \\ A_5^{(n,m)}(x) &\rightarrow A_5^{(n,m)}(x) + i\frac{n}{R_1}\alpha^{(n,m)}(x)\,, \\ A_6^{(n,m)}(x) &\rightarrow A_6^{(n,m)}(x) + i\frac{m}{R_2}\alpha^{(n,m)}(x)\,. \end{aligned} \tag{1.87}$$

Hence, the fields $a^{(n,m)}$ and $A^{(n,m)}$ transform in the following way

$$\begin{aligned} a^{(n,m)}(x) &\rightarrow a^{(n,m)}(x) + iM_{(n,m)}\alpha^{(n,m)}(x)\,, \\ A^{(n,m)}(x) &\rightarrow A^{(n,m)}(x)\,. \end{aligned} \tag{1.88}$$

Now we can fix (one for each element of the infinite tower of gauge bosons) gauge parameters $\alpha^{(n,m)}$ in such way that the $a(n,m)$ is zero:

$$\alpha^{(n,m)} = \frac{i}{M_{(n,m)}} a_{(n,m)} \,. \tag{1.89}$$

This choice gives rise to the following gauge transformation

$$A_\mu^{(n,m)}(x) \to A_\mu^{(n,m)}(x) + \frac{i}{M_{(n,m)}} \partial_\mu a^{(n,m)}(x) \,. \tag{1.90}$$

As a consequence of the transformation in eq.(1.90), the scalar fields $a^{(n,m)}$ are absorbed as longitudinal polarizations of massive gauge bosons and eq. (1.82) takes the form:

$$\begin{aligned}
\mathcal{L}_{4D} &= -\frac{1}{4} F_{\mu\nu}^{(0,0)} F^{\mu\nu}_{(0,0)} \\
&- \frac{1}{2} \sum_{\bar{n},\bar{m}=-\infty}^{\infty} \left[\sum_{\mu>\nu} F_{\mu\nu}^{(-\bar{n},-\bar{m})} F^{\mu\nu}_{(\bar{n},\bar{m})} - M^2_{(\bar{n},\bar{m})} A_\mu^{(-\bar{n},-\bar{m})} A^\mu_{(\bar{n},\bar{m})} \right. \\
&- \left. (\partial_\mu A^{(-\bar{n},-\bar{m})})(\partial^\mu A^{(\bar{n},\bar{m})}) + M^2_{(\bar{n},\bar{m})} A^{(-\bar{n},-\bar{m})} A^{(\bar{n},\bar{m})} \right] \\
&- \frac{1}{2} (\partial_\mu A_5^{(0,0)})(\partial^\mu A^5_{(0,0)}) - \frac{1}{2} (\partial_\mu A_6^{(0,0)})(\partial^\mu A^6_{(0,0)}) \,.
\end{aligned} \tag{1.91}$$

So the effective 4-dimensional Lagrangian in the unitary gauge contains:

- A massless gauge boson $A_\mu^{(0,0)}$.
- A *KK tower* of gauge bosons: the KK modes with $n \neq 0$ and/or $m \neq 0$ have masses $M_{(n,m)}$.
- Two massless real scalars, $A_5^{(0,0)}$ and $A_6^{(0,0)}$.
- A *KK tower* of massive real scalars $A^{(n,m)}$ with square masses $M_{(n,m)}$ and $n \neq 0$ and/or $m \neq 0$.

The only residual gauge invariance of eq.(1.91) is the one related to the only massless gauge boson $A_\mu^{(0,0)}$.

Finally, notice that we can reproduce the effective Lagrangian in eq.(1.91) introducing the $D = 6$ R_ξ-gauge fixing term of type

$$\begin{aligned}
\mathcal{L}^{6D}_{g.f.} &= -\frac{1}{2\xi} (\partial_\mu A^\mu \;+\; \xi \partial_5 A^5 \;+\; \xi \partial_6 A^6)^2 \\
&= -\frac{1}{2\xi} \sum_{n_1,n_2} \left(\partial_\mu A^\mu_{(n_1,n_2)}(x) \;+\; \xi M_{(n_1,n_2)} a^{(n_1,n_2)}(x) \right) e^{2\pi i \frac{n_1}{R_1} y_1} \, e^{2\pi i \frac{n_2}{R_2} y_2} \,,
\end{aligned} \tag{1.92}$$

integrating over the extra dimensions and taking the limit $\xi \to \infty$.

1.2.2 Field theory on manifold with fixed points

In this section, we compactify the non compact manifold $M = \mathbb{R}^d$ using the group of symmetries $G = \Lambda \cup \mathbb{Z}_N$, that is the group of discrete translations by vectors of lattice Λ and d-dimensional rotations θ such that $\theta^N = 1$.

As we have seen in section 1.2, the resulting compact manifold $\mathcal{T}^d/\mathbb{Z}_N$ always contains *singular points*, that is points invariant (up to translations by lattice vectors) under the action of a fixed element $\theta_n \in \mathbb{Z}_N$ with $n = 0, 1, .., N-1$.

We emphasize that the points called *fixed* are *space-time subspace of* $D-d$ *dimensions*, that is in our case these *points* are 4-dimensional subspaces.

The general form of the effective Lagrangian can be parametrized as

$$\mathcal{L}(x, \vec{y}) = \mathcal{L}^{D=4+d}(x, \vec{y}) + \sum_{n=1}^{[N/2]} \sum_{i_n} \delta^{(d)}(\vec{y} - \vec{y}_{i_n}) \mathcal{L}_{4,i_n}(x) \,, \tag{1.93}$$

where $\mathcal{L}^{D=4+d}$ represents the bulk D-dimensional Lagrangian and $\mathcal{L}_{4,i_n}$ the localized lagrangians at fixed points $\vec{y}_{i_n}$. Since $\mathcal{L}$ has to be θ invariant and θ acts non-trivially over some fixed points, there are in general various non trivial constraints among the $\mathcal{L}_{4,i_n}$'s. In addition, the orbifold structure respects a discrete translational symmetry mapping θ fixed points onto θ fixed points[6]. This implies that the lagrangians $\mathcal{L}_{4,i_n}$ are constrained to be all equal at fixed n and hence there are only $[N/2]$ independent localized terms appearing in eq.(1.93).

The physics only depends on the points of the compact space $\mathcal{T}^d/\mathbb{Z}_N$ that is

$$\mathcal{L}(x, \tau_\theta[\vec{y}]) \equiv \mathcal{L}(x, \vec{y}) \,. \tag{1.94}$$

This implies that it is possible to choose a non trivial embedding of the $\mathbb{Z}_N$ orbifold group in the internal space: two fields evaluated in two points connected by an orbifold rotation τ_θ, differ by a transformation O_θ belonging to the (global o local) symmetry group of the Lagrangian

$$\psi(x, \vec{y}) = O_\theta \psi(x, \tau_\theta[\vec{y}]) \;. \tag{1.95}$$

O_θ is the embedding of orbifold rotation in the internal space and then has to satisfy $O_\theta^N = 1$. Now we focus our attention on how the orbifold boundary conditions allow to obtain $D = 4$ chirality.

[6]This is true only in the absence of localized matter that is not uniformly distributed over the fixed points or of discrete Wilson lines

Example: free fermion on orbifold $\mathcal{T}^2/\mathbb{Z}_N$ and chirality

The Lagrangian of a massless $D=6$ Weyl fermion is given by

$$\mathcal{L} = \overline{\Psi}_L i\Gamma^M \partial_M \Psi_L \ . \tag{1.96}$$

where the $D=6$ Dirac matrices Γ^M are defined in eq.(1.28) and the definition of a $D=6$ Weyl fermion in terms of $D=4$ Weyl fermions is given in eq.(1.34).

Now, we compactify the two extra dimensions on the orbifold $\mathcal{T}^2/\mathbb{Z}_N$ with $N = 2,3,4,5$. θ is a generic element of $\mathbb{Z}_N$ and then $\theta^N = 1$. Its eigenvalues are $e^{2\pi i \frac{n}{N}}$ with $n = 0,..,N-1$. For simplicity, we set the two radii of compactifications $R_1 = R_2 = 1$ also in the case of $\mathbb{Z}_N = \mathbb{Z}_2$.

The geometric part of the $\mathbb{Z}_N$ action on a field is fixed by the decomposition of its representation under the 6-dimensional $SO(1,5)$ Lorentz group in terms of $SO(1,3) \times SO(2)$, where $SO(1,3)$ is the 4-dimensional Lorentz group and $SO(2) \simeq U(1)$ is the group of internal rotations (rotations in the two extra dimensions). The $\mathbb{Z}_N$ orbifold boundary conditions of a generic bosonic or fermionic field ϕ with $U(1)$ charge s under internal rotations are then given by

$$\phi(x, \theta\vec{y}) = \eta_{B,F} O_\theta \phi(x, \vec{y}) \ , \tag{1.97}$$

The overall phases $\eta_{B,F}$ are such that $(\eta_B)^N = 1$ for bosons and $(\eta_F)^N = -1$ for fermions, since $O_\theta^N = \pm 1$ in the two cases.

Let's check this statement for the fermionic case. Introduce, first, the following coordinate complex basis for the two extra dimensions

$$\begin{array}{ll} z = \frac{1}{\sqrt{2}}(y_1 + iy_2) & \partial_z = \frac{1}{\sqrt{2}}(\partial_{y_1} - i\partial_{y_2}) \\ \bar{z} = \frac{1}{\sqrt{2}}(y_1 - iy_2) & \partial_{\bar{z}} = \frac{1}{\sqrt{2}}(\partial_{y_1} + i\partial_{y_2}) \end{array} \ . \tag{1.98}$$

In this basis the orbifold action on the coordinates reduces to

$$\tau_\theta = \begin{pmatrix} e^{2\pi i \frac{n}{N}} & 0 \\ 0 & e^{-2\pi i \frac{n}{N}} \end{pmatrix} \ , \tag{1.99}$$

with $n = 0,..,N-1$ whereas the 6-dimensional fermionic Lagrangian reads

$$\mathcal{L} = \overline{\Psi}_L i\Gamma^\mu \partial_\mu \Psi_L + \overline{\Psi}_L i\Gamma^{\bar{z}} \partial_z \Psi_L + \overline{\Psi}_L i\Gamma^z \partial_{\bar{z}} \Psi_L \ . \tag{1.100}$$

Γ^z and $\Gamma^{\bar{z}}$ are, instead, given by

$$\begin{aligned} \Gamma^z &= \gamma^5 \otimes i\frac{\sigma_1 - i\sigma_2}{\sqrt{2}} = \gamma^5 \otimes i\sigma^+ \\ \Gamma^{\bar{z}} &= \gamma^5 \otimes i\frac{\sigma_1 + i\sigma_2}{\sqrt{2}} = \gamma^5 \otimes i\sigma^- \ . \end{aligned} \tag{1.101}$$

Under the orbifold action

$$\begin{aligned} \partial_z &\rightarrow e^{-2\pi i \frac{n}{N}} \partial_z \,, \\ \partial_{\bar{z}} &\rightarrow e^{2\pi i \frac{n}{N}} \partial_{\bar{z}} \,, \end{aligned} \tag{1.102}$$

and then in order to guarantee that the 6-dimensional fermionic Lagrangian is invariant under orbifold transformations, we have to impose

$$\begin{aligned} \overline{\Psi}_L \Gamma^{\bar{z}} \Psi_L &\rightarrow e^{2\pi i \frac{n}{N}} \overline{\Psi}_L \Gamma^{\bar{z}} \Psi_L \,, \\ \overline{\Psi}_L \Gamma^{z} \Psi_L &\rightarrow e^{-2\pi i \frac{n}{N}} \overline{\Psi}_L \Gamma^{z} \Psi_L \,. \end{aligned} \tag{1.103}$$

Hence, we have to find an operator O_θ and an overall phase as in eq.(1.97) satisfying

$$\begin{aligned} \eta^N O_\theta^N &= 1 \\ O_\theta (\gamma^5 \otimes i\sigma^+) O_\theta^\dagger &= e^{2\pi i \frac{n}{N}} (\gamma^5 \otimes i\sigma^+) \\ O_\theta (\gamma^5 \otimes i\sigma^-) O_\theta^\dagger &= e^{-2\pi i \frac{n}{N}} (\gamma^5 \otimes i\sigma^-) \,. \end{aligned} \tag{1.104}$$

A possible solution is

$$\begin{aligned} O_\theta &= e^{2\pi i \frac{n}{N} s \sigma_3} \\ \eta &= e^{2\pi i \frac{n}{N} s\, p} \,, \end{aligned} \tag{1.105}$$

where s is the $U(1)$ charge (seminteger for fermions and integer for bosons) and p is an integer. Finally, the orbifold boundary conditions for a 6-dimensional fermion are given by

$$\Psi(x, e^{2\pi i \frac{n}{N}} z) = e^{2\pi i \frac{n}{N} s \sigma_3} \, e^{2\pi i \frac{n}{N} s\, p} \, \Psi(x, z) \,. \tag{1.106}$$

Furthermore, a spinor living on the compact space $\mathcal{T}^2/\mathbb{Z}_N$ must satisfy the generalized periodicity conditions

$$\begin{aligned} \Psi(x, z + \frac{2\pi}{\sqrt{2}}) &= T_1 \Psi(z) \\ \Psi(x, z + e^{2\pi i \frac{1}{N}} \frac{2\pi}{\sqrt{2}}) &= T_2 \Psi(z) \\ T_{1,2} &\equiv e^{2\pi i \frac{n}{N} t_{1,2}} \,, \end{aligned} \tag{1.107}$$

with $t_{1,2}$ integers. Consistency with the geometric action of translations and rotations requires constraints on the allowed values of the integers $t_{1,2}$ and p (see i.e. [87]). For $\mathcal{T}^2/\mathbb{Z}_2$ one needs $p = \pm 1$, $t_{1,2} = 0, 1$. For $\mathcal{T}^2/\mathbb{Z}_3$ p is in the range $-2, 0, 2$ while $t_1 = t_2 = 0, 1, 2$. In $\mathcal{T}^2/\mathbb{Z}_4$ case, one has $p = \pm 3, \pm 1$ and $t_1 = t_2 = 0, 1$. For $\mathcal{T}^2/\mathbb{Z}_6$, finally, $p = \pm 5, \pm 3, \pm 1$ and $t_1 = t_2 = 0$

Eq.(1.106) can be re-written in terms of 4-dimensional fermions as follows

$$\Psi(x,\theta z) = \begin{pmatrix} \psi_L \\ \chi_R \end{pmatrix}(x,\theta z) = \begin{pmatrix} e^{2\pi i \frac{n}{N} s(1+p)} & 0 \\ 0 & e^{2\pi i \frac{n}{N} s(-1+p)} \end{pmatrix} \begin{pmatrix} \psi_L \\ \chi_R \end{pmatrix}(x,z) \,. \tag{1.108}$$

As we have observed before, the zero modes exist only for fields invariant under the orbifold action. In order to obtain $D = 4$ chirality, it is possible to fix the orbifold boundary conditions in such a way that only one between ψ_L and χ_R is invariant under the orbifold action by choosing a specific value of p. Notice that for a $D = 6$ chiral fermion, regardless of the choice of $\mathbb{Z}_N$, ψ_L and χ_R never admit simultaneously zero modes.

The values of p are restricted by the geometry and, in particular, without considering the embedding of the orbifold action in the internal space, eq.(1.108) shows that in $\mathcal{T}^2/\mathbb{Z}_3$ case we cannot have 4-dimensional chiral zero mode.

1.3 Gauge Symmetry breaking in orbifold compactification

Here, we study different ways of implementing gauge symmetry breaking in the context of field theory on orbifolds. In particular we will concentrate on the relation between boundary conditions and residual symmetries of the effective 4-dimensional theory.

For the next general discussion we use the following notation:

$$\begin{array}{rcl} S_1/\mathbb{Z}_2 & : & z = y_1 \\ \mathcal{T}^2/\mathbb{Z}_N & : & \left\{ \begin{array}{l} z = \frac{1}{\sqrt{2}}(y_1 + iy_2) \\ \bar{z} = \frac{1}{\sqrt{2}}(y_1 - iy_2) \end{array} \right. \end{array} \tag{1.109}$$

$$\begin{array}{rcl} S_1/\mathbb{Z}_2 & : & A_z = A_5 \\ \mathcal{T}^2/\mathbb{Z}_N & : & \left\{ \begin{array}{l} A_z = \frac{1}{\sqrt{2}}(A_5 - iA_6) \\ A_{\bar{z}} = \frac{1}{\sqrt{2}}(A_5 + iA_6) \end{array} \right. \end{array} \tag{1.110}$$

$$\begin{array}{rcl} \theta_n & \in & \mathbb{Z}_N \quad \text{with } n = 0,1,..N-1 \ , \quad \text{and } (\theta_n)^N = 1 \\ \theta & = & \theta_{n=0} \qquad \text{and} \qquad \theta_n = (\theta)^n \ . \end{array} \tag{1.111}$$

The orbifold projection is defined by a geometric action θ representing a symmetry $z \longrightarrow \theta z$ of the internal space S_1 or $\mathcal{T}^2$ and generating the finite group $\mathbb{Z}_N$, which is also embedded into the gauge symmetries of the original theory.

The geometric part of the $\mathbb{Z}_N$ action on the bulk fields Ψ and A_M is fixed by the decomposition of its representation under the symmetry group $SO(1,5)$ $(SO(1,4))$ in terms of $SO(1,3) \times SO(2)$ $(SO(1,3) \times \mathbb{Z}_2)$. $SO(1,3)$ is the 4-dimensional Lorentz group and $SO(2)$ or $\mathbb{Z}_2$ are symmetry groups of extra dimensions in the two cases respectively.

The orbifold projection acts on the gauge group G through an automorphism on its Lie algebra, *i.e.* through a transformation of the type $\lambda^A \to \mathcal{O}^A_B \lambda^B$ that leaves the structure constants of the group invariant. When the automorphism can be written as a group conjugation, $\mathcal{O}^A_B \lambda^B = O\lambda^A O^{-1}$, with $O \in G$, it is called inner automorphism; otherwise it is called an outer automorphism (see [88] for more details). For simplicity, we will restrict to inner automorphisms, where O satisfies $O^N = I$. The components of the gauge fields A_μ with $\mu = 0,1,2,3$ are insensitive to the geometric action of θ and its transformation properties under the orbifold rotations are uniquely given by the matrix O:

$$A_\mu(\theta z) = O A_\mu(z)\, O^{-1} \,. \tag{1.112}$$

The orbifold boundary conditions in eq.(1.112) break the gauge group G to the subgroup H that commutes with O; more precisely, only the fields A^A_μ, associated to H, such that $O\lambda^A O^{-1} = \lambda^A$ are invariant under the orbifold action and admit zero modes, whereas in general the fields $A^{\hat{A}}_\mu$ such that $O\lambda^{\hat{A}} O^{-1} = \mathcal{O}^{\hat{A}}_{\hat{B}} \lambda^{\hat{B}}$ do not admit zero modes.

The general gauge symmetry breaking $G \to H$ is most efficiently described [89, 90] by distinguishing the Cartan generators H_I of the Lie algebra $\mathcal{G}$ associated to G, with $I = 1, \ldots, \operatorname{rank} G$, from the remaining ones E_A, with $A = 1, \ldots, \dim G - \operatorname{rank} G$. The structure of the algebra is then as follows:

$$[H_I, H_J] \;=\; 0 \,, \tag{1.113}$$
$$[H_I, E_A] \;=\; \rho^A_I E_A \,, \tag{1.114}$$
$$[E_A, E_B] \;\propto\; E_{A+B} \,. \tag{1.115}$$

The commutation relation in eq.(1.114) defines the root vector ρ^A_I associated to each E_A, and the commutation relation in eq.(1.115) is vanishing whenever the right-hand side does not exist. Assuming without loss of generality that orbifold embedding in the gauge space O is diagonal, its general form involves only the Cartan generators H_I and is parametrized by a vector V_I as

$$O = e^{2\pi i V_I H_I} \,. \tag{1.116}$$

The vector V_I is constrained by the condition $O^N = I$, but is otherwise arbitrary. The 4-dimensional gauge field can accordingly be decomposed as $A_\mu = A^I_\mu H_I + A^A_\mu E_A$. It follows from the commutation relations in eq.(1.113) and (1.114) that all the modes A^I_μ are even and lead to a zero mode. The modes A^A_μ have boundary conditions twisted by the phase $e^{2\pi i V\cdot\rho_A}$. All the non-Cartan generators for which $V\cdot\rho_A$ is an integer will lead to zero modes and thus to 4-dimensional unbroken symmetries. Thanks to eq.(1.115), all the elements of the group G that admit zero modes form a subgroup $H\subset G$, whose rank coincides with that of G. We emphasize that this type of symmetry breaking is an explicit breaking since the zero modes of the gauge bosons related to the broken symmetries are project out from the 4-dimensional effective theory.

The compact manifold (S_1 or $\mathcal{T}^2$) is in general not simply connected and hence, in addition to the orbifold boundary conditions in eq.(1.112), one has also to specify the periodicity conditions of space-time fields around its non-contractible cycles [45, 46, 89, 91, 92]. Denoting by e_a the basis vectors of the non-contractible cycles γ_a in the compact manifold, one can impose for A_μ a general boundary condition that is twisted through arbitrary matrices W_a of the gauge group G in the fundamental representation:

$$A_\mu(z+e_a)=W_a A_\mu(z)\,W_a^{-1}\,. \tag{1.117}$$

The twist matrices W_a can be interpreted [60–62] as Wilson lines along the cycles γ_a:

$$W_a=\mathcal{P}\,\exp i\oint_{\gamma_a}A\,, \tag{1.118}$$

where $\mathcal{P}$ denotes the usual path ordering. Only a subset of the Wilson lines that are allowed on the compact manifold (S_1 or $\mathcal{T}^2$) gives well-defined Wilson lines on $S_1/\mathbb{Z}_2$ or $\mathcal{T}^2/\mathbb{Z}_N$. The precise consistency conditions depend on the explicit form of compact manifold and must be discussed case by case. A general feature distinguishing the solutions to the Wilson line consistency conditions is that they may or may not depend on continuous parameters. The first are called continuous and the other discrete Wilson lines. We will do this in some detail for the simplest 1- and 2-dimensional orbifold constructions, in which the Wilson lines in eq.(1.118) arise from constant connections and so the path ordering is irrelevant.

Wilson lines represent an additional possibility for gauge symmetry breaking, since only the gauge fields A_μ left unbroken by the projection O and periodic around all the cycles of the internal space admit 4-dimensional massless modes. The combined gauge symmetry breaking due to the boundary conditions in eq.(1.112) and eq.(1.117) can be alternatively understood in terms of the local effective symmetry at the various fixed points of the orbifold action. The crucial property allowing this reinterpretation is that a generic fixed point p_{i_n} is left fixed by the element θ_n only modulo a suitable translation in the internal space as we have seen in eq.(1.51). Combining eq.(1.112), eq.(1.117) and eq.(1.51), we deduce that at a given θ_n fixed point p_{i_n}, with associated integers $q_{i_n\,a}$, the effective orbifold projection is implemented by a matrix that is not just O_n but rather

$$O_{i_n}=\prod_a W_a^{q_{i_n\,a}}O_n\,. \tag{1.119}$$

More precisely, this means that only those components of the gauge field that commute with O_{i_n} can possibly have zero modes. The gauge group G is therefore locally broken at $z = p_{i_n}$ to the subgroup H_{i_n} of G commuting with O_{i_n} at $z = p_{i_n}$. The globally unbroken gauge group H in 4 dimensions is then the intersection H of the gauge groups surviving at all the fixed points: $H = \cap_{i_n} H_{i_n}$. Depending on whether the Wilson lines W_a commute or not with the projection O, rank-preserving or rank-reducing gauge symmetry breaking are possible.

1.3.1 $S^1/\mathbb{Z}_2$

Consider a bulk Dirac fermion field Ψ in an arbitrary representation r of the gauge group G in interaction with the gauge fields. The action of the reflection on the fields depends on the orbifold matrix O and on an overall sign choice for the fermion which can in general depend on the representation r. The orbifold boundary conditions then read:

$$
\begin{aligned}
\Psi(-z) &= \pm\gamma_5 O_r \Psi(z)\,, \\
A_\mu(-z) &= O A_\mu(z) O^{-1}\,, \\
A_z(-z) &= -\,O A_z(z) O^{-1}\,,
\end{aligned}
\tag{1.120}
$$

In these expressions, $O = O_{\text{fund}}$ and O_r are matrices in the fundamental representation and in the generic representation r, respectively. Since S_1 is not simply connected, we also need to specify the corresponding periodicity conditions. These are in general twisted by a matrix W, and read

$$
\begin{aligned}
\Psi(z + 2\pi R) &= W_r \Psi(z)\,, \\
A_M(z + 2\pi R) &= W A_M W^{-1}(z)\,,
\end{aligned}
\tag{1.121}
$$

where $W = W_{\text{fund}}$ and W_r represent the Wilson line in the fundamental representation and in the generic representation r.

The fixed points $p_0 = 0$ and $p_1 = \pi R$ have $q_0 = 0$ and $q_1 = 1$ in eq.(1.51), and the corresponding effective projections are $O_0 = O$ and $O_1 = WO$. Denoting by H_1 and H_2 the associated gauge subgroups, the surviving gauge group in 4 dimensions is $H = H_1 \cap H_2$.

The gauge twist can be interpreted as a Wilson line

$$
W = \exp\{i \oint \langle A_z\rangle\} = \exp\{2\pi R \langle A_z\rangle\}\,,
\tag{1.122}
$$

constructed from a non-vanishing $\langle A_z\rangle$ that is constant and compatible with the boundary conditions for A_z. This is possible as a consequence of the fact that $\langle A_z\rangle$ does not necessarily commute with O and is moreover defined only up to the equivalence class

$\langle A_z \rangle = \langle A_z \rangle + p/R$, where p is any integer, dictated by periodic gauge transformations on S^1 (see *i.e.* [93]). The allowed values for W can be determined by noting that the geometrical actions T and θ satisfy the relation $(\theta T)^2 = I$. As a consequence, the generic boundary conditions eq.(1.120) and eq.(1.121) on the fields are mutually consistent only if the corresponding twist matrices O and W satisfy the relation [45, 46, 94, 95]

$$(OW)^2 = I\,. \tag{1.123}$$

Two possibilities can then arise, depending on whether the Wilson line originates from an even or an odd component of A_z. The generators λ^A of the Lie algebra of G that correspond to components of A_z that are even under both projections effectively implemented at the two fixed points, and therefore lead to zero modes for A_z, are specified, as a consequence of eq.(1.123) and eq.(1.120), by the following two conditions:

$$\left\{O, \lambda^A\right\} = 0\,, \quad \left\{WO, \lambda^A\right\} = 0\,. \tag{1.124}$$

Together, these also imply that $[W, \lambda^A] = 0$. These Wilson lines are continuous, since the even fields A_z^A from which they are constructed can take an arbitrary constant vacuum expectation values. Recall that no potential for the 4-dimensional scalar fields A_z is allowed by gauge invariance on the orbifold $S^1/\mathbb{Z}_2$ and thus A_z^A are moduli fields, at least at tree level[7]. Due to the first of the conditions in eq.(1.124), W cannot commute with O, $[O, W] \neq 0$, so that continuous Wilson lines typically induce a spontaneous rank-reducing gauge symmetry breaking.

On the other hand, the generators $\lambda^{\hat{A}}$ that correspond to components of A_z that are odd under both local orbifold projections, and therefore do not lead to zero modes for A_z, are specified by the conditions:

$$\left[O, \lambda^{\hat{A}}\right] = 0\,, \quad \left[WO, \lambda^{\hat{A}}\right] = 0\,. \tag{1.125}$$

As before, these imply that $[W, \lambda^{\hat{A}}] = 0$. In this case, the Wilson lines constructed from the corresponding $A_z^{\hat{A}}$ are discrete. Indeed, only the two specific values $\langle A_z^{\hat{A}} \rangle = 0$ and $1/(2R)$ satisfy the odd orbifold boundary condition about each fixed point, thanks to the fact that a shift by $1/R$ in $\langle A_z^{\hat{A}} \rangle$ is irrelevant. In this case W commutes with O, $[O, W] = 0$, so that discrete Wilson lines induce a rank-preserving gauge symmetry breaking. This can also be understood from the fact that the orbifold projection acts with opposite signs on A_μ and A_z; the gauge fields $A_\mu^{\hat{A}}$ are therefore even under both O and WO and the generators $\lambda^{\hat{A}}$ correspond to the unbroken gauge group H in 4 dimensions. This is not a spontaneous symmetry breaking, but rather a truncation and as such it is more similar to an orbifold projection.

[7] We illustrate in chapter 5 the standard technique to calculate its values at one loop.

Finally, it can easily be verified that the remaining components of A_z, which are even under one of the local projections and odd under the other, can never give rise to consistent Wilson lines on $S^1/\mathbb{Z}_2$.

It is worth mentioning that the orbifold $S^1/\mathbb{Z}_2$ in the presence of a discrete Wilson line W can be equivalently described in terms of another orbifold, constructed from a circle $S^{1\prime}$ of radius $R' = 2R$ that is the double cover of the original S^1. Since $W^2 = I$, all fields are periodic around $S^{1\prime}$, and the projection $O' = WO$ is now realized through a new independent $\mathbb{Z}_2'$ reflection that is orthogonal to the original $\mathbb{Z}_2$ reflection and acts as inversion around the point $\pi R'/2$ of the circle. The resulting space is thus an $S'^1/(\mathbb{Z}_2\times\mathbb{Z}_2')$ orbifold.

We conclude this subsection emphasizing that, in addition to the symmetries H, there is another residual symmetry at fixed points that is of crucial importance in the building of realistic models. In fact, at the fixed points $p_0 = 0$ and $p_1 = \pi R$, there is a symmetry acting in a non linear way on A_z [28, 66]:

$$\delta A_z = \partial_5 \xi \,, \tag{1.126}$$

where ξ are the gauge parameters of the G/H transformations. In this way, the original bulk gauge symmetry forbids bulk mass terms for A_z and the local residual symmetry at fixed points, forbids localized mass terms for A_z.

Example: Orbifold and Wilson line symmetry breaking:
$SU(3) \to SU(2) \otimes U(1) \to U(1)$

Consider a $SU(3)$ invariant five dimensional theory with a fermion ψ in the fundamental representation of gauge group:

$$\psi = \begin{pmatrix} u \\ d \\ \chi \end{pmatrix} \,. \tag{1.127}$$

The fifth dimension is compactified on the orbifold $S_1/\mathbb{Z}_2$.

In this example we use the result of section 1.2 and describe the geometric properties of orbifold $S_1/\mathbb{Z}_2$ using the orbifold reflection at the fixed point $y = 0$ and the translation T of $2\pi R$. This implies that we have to specify two different boundary conditions: orbifold projection and periodicity conditions. In particular we will use the orbifold projection to reduce (explicitly) $G = SU(3) \to H = SU(2) \otimes U(1)$ and then the Wilson line appearing in the periodicity conditions to (spontaneously) break $H = SU(2) \otimes U(1) \to U(1)$.

The choice of the orbifold embedding in the $SU(3)$ space is strongly constrained. Working with the fundamental representations of $SU(3)$, the matrix O in eq.(1.120) has to be a 3×3 matrix satisfying the following conditions

- $O^2 = \mathbf{1}$.
- $[O, \lambda^a] = 0$, where $a = 1, 2, 3, 8$.
- $O = \; e^{i\frac{\vec{\alpha}\cdot\vec{\lambda}}{2}}$. The orbifold projection acts on the algebra of the $SU(3)$ gauge group as an inner automorphism. It is possible to verify that by imposing the fist two constraints, the third one is equivalent to $\{O, \lambda^{\hat{a}}\} = 0$ with $\hat{a} = 4, 5, 6, 7$.

$\frac{\lambda^a}{2}$ are the $SU(3)$ generators and the λ^a are the Gell-Mann matrices.

The general form of O that satisfies all the constraints is

$$O \;=\; \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{1.128}$$

There are different ways to parametrize this matrix in terms of the $SU(3)$ elements. For instance, choosing the minus sign in eq.(1.128), we have $O = e^{i\pi\lambda_3}$.

The orbifold boundary conditions, therefore, take the following form:

$$\begin{cases} \psi_i(x,-y) \;=\; [\gamma_5 \otimes O_{ij}]\,\psi_j(x,y) = \left[\gamma_5 \otimes \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{pmatrix} u \\ d \\ \chi \end{pmatrix} \\ A_\mu(x,-y) \;=\; A^a_\mu(x,y) O T^a O \\ A_5(x,-y) \;=\; -A^a_5(x,y) O T^a O \end{cases} . \tag{1.129}$$

We have built O in such a way that it commutes with λ^a and anticommutes with $\lambda^{\hat{a}}$. The fields A^a_μ and $A^{\hat{a}}_5$ are even under $\mathbb{Z}_2$, whereas $A^{\hat{a}}_\mu$ and A^a_5 are odd. Only the even ones admit zero modes and survive at the fixed points $y = 0$ and $y = \pi R$.

The original symmetry group $G = SU(3)$ is *explicitly* reduced to $H = SU(2) \otimes U(1)$ at fixed points, since the $SU(3)/(SU(2) \times U(1))$ gauge bosons are projected out by orbifold boundary conditions.

The rank of the effective symmetry group H is the same as the initial one, G: the orbifold projection, in fact, acts on the algebra of $G = SU(3)$ as an inner automorphism.

The Wilson line W has to satisfy the consistency condition in eq. (1.123). We are interested in lowering the rank of the initial symmetry group. This implies that the Wilson line has to be a function only of the $SU(3)$ generators that are odd under the orbifold projection. The most general form of W compatible with the orbifold choice in eq.(1.128) and with the constraint in eq.(1.123) is

$$W = e^{\pi i \vec{\alpha}\cdot\vec{\lambda}} = e^{\pi i(\alpha_4\lambda_4 + \alpha_5\lambda_5 + \alpha_6\lambda_6 + \alpha_7\lambda_7)} . \tag{1.130}$$

Since such Wilson line W has no-trivial transformation properties under the gauge symmetry group $H = SU(2) \otimes U(1)$ compatible with the orbifold projection, it is possible fix the gauge of H in such a way to orient the Wilson line along only one generator, for example λ_7.

The residual symmetries of the effective 4-dimensional theory are associated to those gauge fields A_μ invariant under both orbifold and periodicity conditions, that is to such fields which admit zero modes. In our case, the only zero mode is associated to a linear combination of A^3_μ and A^8_μ giving rise to the symmetry breaking $SU(3) \to SU(2) \otimes U(1) \to U(1)$.

Now, we want to give an argument in order to convince the reader that the second step of the symmetry breaking can be interpreted (from 4-dimensional point of view) as a spontaneous symmetry breaking: *i.e.* a symmetry breaking induced by a non-trivial vev of a 4-dimensional scalar.

The periodicity conditions read

$$\left\{ \begin{array}{lcl} \psi(x, y + 2\pi R) & = & W\psi(x, y) \\ A_M(x, y + 2\pi R) & = & W A_M(x, y) W^{-1} \end{array} \right. . \tag{1.131}$$

A possible choice for fields satisfying eq.(1.131), is

$$\left\{ \begin{array}{lcl} \psi(x, y) & = & e^{i(\vec{\alpha}\vec{\lambda})\frac{y}{2R}} \tilde{\psi}(x, y) \\ A_M(x, y) & = & e^{i(\vec{\alpha}\vec{\lambda})\frac{y}{2R}} \tilde{A_M}(x, y) e^{-i(\vec{\alpha}\vec{\lambda})\frac{y}{2R}} \end{array} \right. , \tag{1.132}$$

where the fields $\tilde{\psi}$ and $\tilde{A_M}$ are fields strictly periodic under translations of $2\pi R$ along y.

Using the symmetries of the non-compactified theory, it is possible to go in the background gauge in which the periodicity conditions in eq.(1.131) become trivial boundary conditions (see chapter 4 for a general discussion of this topic). The non-periodic transformation that allows this background gauge change is the following:

$$\left\{ \begin{array}{lcl} \psi(x, y) & \to & e^{-i(\vec{\alpha}\vec{\lambda})\frac{y}{2R}} \psi(x, y) = \tilde{\psi}(x, y) \\ & & \\ A_M(x, y) & \to & e^{-i(\vec{\alpha}\vec{\lambda})\frac{y}{2R}} A_M(x, y) e^{i(\vec{\alpha}\vec{\lambda})\frac{y}{2R}} - \frac{i}{g} e^{-i(\vec{\alpha}\vec{\lambda})\frac{y}{2R}} \partial_M e^{i(\vec{\alpha}\vec{\lambda})\frac{y}{2R}} \\ & & \\ & = & \tilde{A}_M(x, y) + \frac{(\vec{\alpha}\vec{\lambda})}{2gR} \delta_{M,5} \end{array} \right. . \tag{1.133}$$

Since the Wilson line satisfies the consistency condition in eq.(1.123), the orbifold boundary conditions in eq.(1.129) are formally invariant under this transformation up to the substitution of fields ψ and A_M with $\tilde{\psi}$ and $\tilde{A}_M$ respectively.

In the background gauge defined in eq.(1.133), all fields are periodic and the 4-dimensional scalar A_5 takes a non vanishing VEV. This result explains because in the case of Wilson line symmetry breaking, it is possible to interpret the symmetry breaking as a spontaneous symmetry breaking. Notice that since S^1 is a non-simply connected space, such constant background has physical meaning (it cannot be gauged away!) and should be computed at quantum level [60–62]: see chapter 5.

1.3.2 $\mathcal{T}^2/\mathbb{Z}_N$

The $\mathbb{Z}_N$ projection is embedded as usual in the gauge group through an arbitrary matrix O of G satisfying $O^N = I$.

We consider a 6-dimensional complex fermion Ψ of chirality ρ in an arbitrary representation r of the gauge group G and in interaction with external gauge. The action of the $\mathbb{Z}_N$ rotation on the spinor indices is specified by the $SO(2) \simeq U(1)$ representation under the group of internal rotations. For a fermion of spin 1/2, this is a phase θ^s, where $s = \pm 1/2$ defines the two 4-dimensional components with chirality ± 1. The orbifold action on the gauge degrees of freedom is implemented by the matrix O and can involve a phase η of the form $\eta = \theta^{1/2+r_\eta}$, with $r_\eta = 0, 1, \ldots, N-1$.[8] One then has

$$\begin{aligned}
\Psi(\theta z) &= \eta\, \theta^s O_r \Psi(z)\,, \\
A_\mu(\theta z) &= O\, A_\mu(z)\, O^{-1}\,, \\
A_z(\theta z) &= \theta^{-1}\, O\, A_z(z)\, O^{-1}\,, \\
A_{\bar{z}}(\theta z) &= \theta\, O\, A_{\bar{z}}(z)\, O^{-1}\,.
\end{aligned} \tag{1.134}$$

Similarly, the actions of translations around the two independent cycles of $\mathcal{T}^2$ are encoded in boundary conditions that are in general twisted by two Wilson lines W_1 and W_2 of the gauge group G:

$$\begin{aligned}
\Psi(z + e_a) &= W_{a,r} \Psi(z)\,, \\
A_M(z + e_a) &= W_a A_M(z)\, W_a^{-1}\,,
\end{aligned} \tag{1.135}$$

where $W = W_{fund}$ and W_r represent the Wilson line in the fundamental representation and in the generic representation r. The Wilson lines are specified by the possible constant connections that can exist around the two independent cycles γ_a specified by the basis vectors e_a: $W_a = \exp i\{e_a \langle A_z \rangle + \text{c.c.}\}$. Exactly as in the $S^1/\mathbb{Z}_2$ case, the constant background value $\langle A_z \rangle$ can be compatible with the orbifold boundary conditions, thanks to the equivalence relation $\langle A_z \rangle = \langle A_z \rangle + 2\pi p_a / e_a$ on $\mathcal{T}^2$, with p_a two arbitrary integers. The consistency conditions constraining the matrices O and W_a are in this case quite severe. Indeed, the geometric actions of the orbifold rotations θ and the translations T_a satisfy the relations $(T_1^{q^1_{i_n}} T_2^{q^2_{i_n}} \theta^n)^{N/n} = I$, where $q^1_{i_n}$ and $q^2_{i_n}$ are integer numbers, for each integer $n = 1, \ldots, N/2$ such that $\mathbb{Z}_{N/n}$ is a subgroup of $\mathbb{Z}_N$, *i.e.* N/n is integer, and

[8] Notice that this phase is actually necessary to have a $\mathbb{Z}_N$ action with $\theta^N = 1$ on all fields.

$[T_1, T_2] = 0$. These imply the conditions

$$(W_1^{q^1_{i_n}} W_2^{q^2_{i_n}} O^n)^{N/n} = I\,, \qquad\qquad [W_1, W_2] = 0\,. \tag{1.136}$$

There is an additional condition depending on how the basis vectors e_a are mapped within each other by the rotation. For $N = 2$, each e_a is reflected to $-e_a$, and thus $\theta T_a = T_a^{-1}\theta$, but this does not lead to any new condition. For $N = 3, 4, 6$, since $U = \theta = e^{2\pi i/N}$, one has $\theta e_1 = e_2$, and hence $\theta T_1 = T_2 \theta$. This leads to the condition

$$W_1 O = O W_2\,, \quad \text{for } N = 3, 4, 6\,. \tag{1.137}$$

There can be continuous Wilson lines with $[W_a, O] \neq 0$, associated to a constant connection of a field with a massless mode, or discrete ones, with $[W_a, O] = 0$, where the constant connection corresponds to a discrete deformation of the model. The case of continous Wilson lines is nowadays under our study and here we concentrate on discrete Wilson lines. Since these commute with O, they have to satisfy the relation $(W_1^{q^1_{i_n}})^{N/n}(W_2^{q^2_{i_n}})^{N/n} = I$ for each n. The above conditions leave the following possibilities for discrete Wilson lines in the various models. For $N = 2$, the two Wilson lines W_a are independent and satisfy $W_a^2 = I$. For $N = 3, 4, 6$, they are instead identified by the condition eq.(1.137), $W_1 = W_2 = W$, and satisfy respectively $W^3 = I, W^2 = I, W = I$. In other words, there can be two independent $\mathbb{Z}_2$ Wilson lines in the $\mathbb{Z}_2$ model, a $\mathbb{Z}_3$ Wilson line in the $\mathbb{Z}_3$ model, a $\mathbb{Z}_2$ Wilson line in the $\mathbb{Z}_4$ model, and no Wilson lines at all in the $\mathbb{Z}_6$ model. As before, the presence of discrete Wilson lines induces a distinction between the projections occurring at the various fixed points p_{i_n} in a given sector n, depending on the numbers $q^1_{i_n}$ and $q^2_{i_n}$ of T_1 and T_2 translations that are needed to relate p_{i_n} and its image $\theta_n p_{i_n}$. As a consequence of the presence of the discrete Wilson lines, the effective $\mathbb{Z}_N$ projection at each fixed point p_{i_n}[9] will then involve the matrix

$$O_{i_n} = W_1^{q^1_{i_n}} W_2^{q^2_{i_n}} O^k\,. \tag{1.138}$$

Again, it is interesting to notice that a $\mathcal{T}^2/\mathbb{Z}_N$ orbifold model with a $\mathbb{Z}'_N$ discrete Wilson line can be equivalently understood as a freely acting orbifold of the type $\mathcal{T}^2/(\mathbb{Z}_N \times \mathbb{Z}'_N)$. In this case, a precise map between the two constructions is more difficult to define, because of the non-trivial complex structure of the $\mathcal{T}^2$.

We conclude observing that like the $S_1/\mathbb{Z}_2$ case, the bulk gauge invariance forbids bulk mass terms for the 4-dimensional scalar A_z, *but now it is not possible to find at fixed points any residual symmetry that forbids localized mass terms for* A_z [28, 67].

[9]Notice that some of the points p_{i_n} coincide, since the same fixed point p_0 is generally fixed under more elements of the orbifold action.

Chapter 2

Gauge Flux Compactification

In this chapter we study the compactification of extra dimensions on a torus, in the presence of a magnetic flux: *i.e.* we consider a manifold without fixed points in which an abelian gauge background with constant field strength lives. It is known [48] that this scenario should generically result in 4-dimensional fermion chirality. Such type of construction, therefore, represents an alternative to the orbifold construction in the attempt of realistic model building in the context of extra dimensions.

In this chapter, we will review the abelian case (for a pedagogical introduction see for instance [96]). The analysis of more complicate gauge group, as well as the symmetry breaking patterns that can be realized in this type of construction, will be the argument of subsequent chapters.

2.1 Abelian gauge theory on a torus

Consider a 6-dimensional $U(1)$ gauge theory with a massless Weyl fermion of charge q, on a $\mathcal{M}_4 \times \mathcal{T}^2$ space-time.

$$\mathcal{L}_{6D} = -\frac{1}{4} F_{MN} F^{MN} + i \bar{\Psi}_L \Gamma^M D_M \Psi_L \,, \tag{2.1}$$

where

$$\begin{aligned} D_M &= \partial_M - iqA_M \,, \\ F_{MN} &= \partial_M A_N - \partial_N A_M \,. \end{aligned} \tag{2.2}$$

Consider for simplicity an orthogonal torus of area $\mathcal{A} = l_1 l_2$. Since the space is multiply connected, one has to specify the periodicity conditions of A_M and Ψ around non-contractible cycles. The possible presence of a background should manifest itself in the explicit form of these conditions. In the abelian case, the most general form of the periodicity conditions is

$$\begin{aligned} A_M(x, \vec{y} + l_a) &= A_M(x, \vec{y}) + \partial_M \beta_a(y_1, y_2)\,, \\ \Psi(x, \vec{y} + l_a) &= e^{iq\beta_a(y_1,y_2)}\,\Psi(x, \vec{y})\,, \end{aligned} \tag{2.3}$$

where $a = 1, 2$. After translations, in fact, fields need to return to their original values up to, at most, a gauge transformation.

In order to preserve the 4-dimensional Poincaré invariance, the gauge parameters which appear in the periodicity conditions can only depend on extra coordinates y_1 and y_2. The phases

$$\Omega_a(\vec{y}) \equiv \Omega_a(y_1, y_2) = e^{iq\beta_a(y_1,y_2)}\,, \tag{2.4}$$

are the embeddings of translations $T_a : \vec{y} \to \vec{y} + l_a$ in the gauge space and are called *transition functions*. They can be parametrized as follows

$$\Omega_a(\vec{y}) = exp\left\{-iq\int_{l_a} B_i(\vec{y})dy^i\right\}\,, \tag{2.5}$$

where the index i runs only over the extra dimensions[1]. $B_i(y)$ is a generic periodic abelian background preserving 4-dimensional Poincaré invariance and living on the torus. The transition functions Ω_a must be strictly periodic under translations along the same direction a. Therefore, it is always possible to parametrize Ω_1 and Ω_2 as

$$\begin{aligned} \Omega_1(\vec{y}) &= exp\left\{-iq\int_{l_1} B_1(\vec{y})dy^1\right\}\,, \\ \Omega_2(\vec{y}) &= exp\left\{-iq\int_{l_2} B_2(\vec{y})dy^2\right\}\,, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} B_1(\vec{y}) &= \sum_{n=-\infty}^{\infty} B_1^{(n)}(y_2)e^{\frac{2\pi i n}{l_1}y_1}\,, \\ B_2(\vec{y}) &= \sum_{m=-\infty}^{\infty} B_2^{(n)}(y_1)e^{\frac{2\pi i m}{l_2}y_2}\,. \end{aligned} \tag{2.7}$$

Substituting eq.(2.7) into eq.(2.6), it follows that

$$\begin{aligned} \Omega_1(\vec{y}) &= exp\left\{-iq\int_{l_1} B_1^{(0)}(y_2)dy^1\right\} = exp\left\{-iqB_1^{(0)}(y_2)l_1\right\}\,, \\ \Omega_2(\vec{y}) &= exp\left\{-iq\int_{l_2} B_2^{(0)}(y_1)dy^2\right\} = exp\left\{-iqB_2^{(0)}(y_1)l_2\right\}\,. \end{aligned} \tag{2.8}$$

[1] In this chapter, we will use the typical notation of the literature about flux compactification, also for quantities already introduced in previous chapters.

Comparing the result in eq.(2.8) with the expression in eq.(2.4), it is possible to deduce a relation between the phases β_a and the abelian backgrounds B_a:

$$\begin{aligned}\beta_1(y_1, y_2) &= -l_1\, B_1^{(0)}(y_2)\,,\\ \beta_2(y_1, y_2) &= -l_2\, B_2^{(0)}(y_1)\,.\end{aligned} \tag{2.9}$$

The following step consists in determining the possible abelian backgrounds that can live on a torus. To do that, we recall that the transition functions Ω_a are the embeddings of translations in the gauge space. Translations by the full length of the torus, that is around the two independent non-contractible cycles, must obviously commute,

$$[T_1, T_2] = 0\,, \tag{2.10}$$

resulting in an interesting constraint for the transition functions [68]

$$\Omega_1(\vec{y} + l_2)\Omega_2(\vec{y}) = \Omega_2(\vec{y} + l_1)\Omega_1(\vec{y}). \tag{2.11}$$

Substituting, now, the expression in eq.(2.8) in the constraint of eq.(2.11), it follows that

$$e^{iq\int_{\mathcal{A}_{T^2}} G_{12} d\mathcal{A}} = 1\ , \tag{2.12}$$

where

$$G_{12} = \partial_1 B_2^{(0)} - \partial_2 B_1^{(0)}\,. \tag{2.13}$$

The $U(1)$ gauge background that can live on a torus, therefore, has to satisfy the following constraint on its flux

$$\int_{\mathcal{A}_{T^2}} G_{12} d\mathcal{A} = 2\pi m/q\,, \tag{2.14}$$

where m is an arbitrary integer number and q is the fermion charge. In the abelian case, m is the first Chern number.

Notice that the result in eq.(2.14) strongly constrains the possible $U(1)$ charges which can live on a torus. In fact, for two fields with different $U(1)$ charges q_1 and q_2, we obtain at the same time the following quantization conditions

$$e^{iq_1\int_{\mathcal{A}_{T^2}} G_{12} d\mathcal{A}} = 1\ , \qquad\qquad e^{iq_2\int_{\mathcal{A}_{T^2}} G_{12} d\mathcal{A}} = 1\,. \tag{2.15}$$

When $m \neq 0$, such system admits solution only when the two charges are integer multiple of a fundamental $U(1)$ charge: **all $U(1)$ charges must be quantized in terms of a fundamental charge**.

A possible solution of eq.(2.12) is given by

$$\begin{array}{rcl} B_1^{(0)}(y_2) = -\frac{\pi m}{q l_1 l_2} y_2 & \rightarrow & \Omega_1 = e^{i\pi m \frac{y_2}{l_2}} \\ B_2^{(0)}(y_1) = \frac{\pi m}{q l_1 l_2} y_1 & \rightarrow & \Omega_2 = e^{-i\pi m \frac{y_1}{l_1}} \,. \end{array} \tag{2.16}$$

The abelian background in eq.(2.16) is called *magnetic background* since it gives rise to a constant (space-like) field strength:

$$G_{12} = \partial_1 B_2^{(0)} - \partial_2 B_1^{(0)} = \frac{2\pi m}{q l_1 l_2} \,. \tag{2.17}$$

Summarizing, the periodicity conditions for fields living on a torus in presence of a quantized $U(1)$ magnetic flux (up to a gauge background change) read

$$\begin{aligned} A_M(x, \vec{y} + l_a) &= A_M(x, \vec{y}) + \frac{1}{q} \partial_M \left(\epsilon_{ab} \frac{\pi m}{l_b} y_b \right) , \\ \Psi(x, \vec{y} + l_a) &= e^{i \epsilon_{ab} \pi m \frac{y_b}{l_b}} \Psi(x, \vec{y}) \,, \end{aligned} \tag{2.18}$$

where ϵ_{ab} is the antisymmetric tensor with two indices ($\epsilon_{12} = 1$). Fields satisfying these boundary conditions span a Hilbert space $\mathcal{H}_m$ with scalar product

$$\langle \phi | \psi \rangle = \int_{\mathcal{A}_T^2} d\vec{y} \phi^*(\vec{y}) \psi(\vec{y}) \,, \tag{2.19}$$

where the integration is over a fundamental cell of the torus.

The generators of ordinary translations in presence of a background with constant field strength are given by

$$D_a = \partial_a + i \frac{\pi m}{l_1 l_2} \epsilon_{ab} \; y_b \,, \tag{2.20}$$

The algebra of such generators is simply

$$[D_1, D_2] = -i G_{12} = -i \frac{2\pi m}{l_1 l_2} \,. \tag{2.21}$$

Translations by an arbitrary length, h_a, instead, are given by

$$\begin{aligned} \mathcal{T}_1(h_1) \Psi(x, \vec{y}) &= e^{i\pi m \frac{h_1 y_2}{l_1 l_2}} \Psi(x, \vec{y} + h_1) \,, \\ \mathcal{T}_2(h_2) \Psi(x, \vec{y}) &= e^{-i\pi m \frac{h_2 y_1}{l_1 l_2}} \Psi(x, \vec{y} + h_2) \,. \end{aligned} \tag{2.22}$$

In presence of magnetic background two translations of an arbitrary quantity along the two independent axis do not commute:

$$\mathcal{T}_1(h_1) \, \mathcal{T}_2(h_2) \; = \; e^{-2\pi i m \frac{h_1 h_2}{l_1 l_2}} \; \mathcal{T}_2(h_2) \, \mathcal{T}_1(h_1) \,. \tag{2.23}$$

The non-commutativity arises from the presence of a non-zero constant magnetic background.

Finally, the periodicity conditions in eq.(2.18) have a another symmetry which will play a fundamental role in the analysis of wave functions. It can be shown [96], indeed, that there is a group of independent unitary transformations which commute with ordinary translations and do leave the torus invariant. Their generators $K^{1,2}$ can be denoted by

$$\begin{aligned} K^1 &= exp\left\{2\pi i\frac{y_1}{l_1}\right\}\mathcal{T}_2(\frac{l_2}{m}) \\ K^2 &= exp\left\{2\pi i\frac{y_2}{l_2}\right\}\mathcal{T}_1(-\frac{l_1}{m})\,. \end{aligned} \tag{2.24}$$

Using boundary conditions, it is easy to prove that the transformations K^1 and K^2 satisfy

$$\begin{aligned} \left(K^1\right)^m &= \left(K^2\right)^m = \mathbb{I} \\ K^1K^2 &= e^{-2\pi i/m}K^2K^1\,. \end{aligned} \tag{2.25}$$

Any of the K generates, therefore, a $\mathbb{Z}_m$ group under which $\mathcal{H}_m$ decomposes into m orthogonal subspaces:

$$\mathcal{H}_m = \oplus_{n=1}^{m}\mathcal{H}_{m,n}\,. \tag{2.26}$$

It is then possible to catalogue the states of the Hilbert space by two quantum numbers, m and the eigenvalue of one of K's. If, for instance, we choose K^2, that is a basis of the Hilbert space satisfying

$$K^2\Psi(x,\vec{y}) = e^{-2\pi i\frac{n}{m}}\Psi(x,\vec{y})\,, \tag{2.27}$$

the operator K^1 maps the subspaces $\mathcal{H}_{m,n}$ into each other, whereas non-commutative translations leave the subspaces invariant because they commute with $K^{1,2}$. This result has the important consequence that, for any fields which satisfy the periodic conditions in eq.(2.18), there exists m "replicas", one for each subspace $\mathcal{H}_{m,n}$. This result will be useful for the discussion of following sections.

2.2 Fermions and the chirality problem

Consider a 6-dimensional chiral fermion, Ψ_L, in presence of a generic $U(1)$ background. Before analyzing in details the case of magnetic background, we are interested in deducing the minimal set of properties that a background living on a torus must have in order to obtain 4-dimensional chirality.

To preserve $D = 4$ Lorentz invariance, the gauge background is set to live only on the extra-dimensions. The equations of motion for a $D = 6$ Weyl fermion in this setup read

$$\Gamma^\mu \partial_\mu \Psi + i\Gamma^1 D_1 \Psi + i\Gamma^2 D_2 \Psi = 0 \;, \tag{2.28}$$

where $D_{1,2} = \partial_{1,2} - iqB_{1,2}$ and the Γ matrices are those of eq.(1.28) with the convention $\Gamma^1 = \Gamma^4$ and $\Gamma^2 = \Gamma^5$. Recall the expression of a $D = 6$ Weyl spinor in terms of two $D = 4$ chiral spinors

$$\Psi_L = \begin{pmatrix} \psi_L \\ \chi_R \end{pmatrix} \;, \tag{2.29}$$

and introduce the new basis

$$\begin{cases} D_{\bar{z}} = \frac{1}{\sqrt{2}}(D_1 + iD_2) \\ D_z = \frac{1}{\sqrt{2}}(D_1 - iD_2) \end{cases} . \tag{2.30}$$

Using eq.(1.28), the equations of motion can be written as

$$\begin{cases} i\gamma^\mu \partial_\mu \psi_L - \gamma_5 D_{\bar{z}} \chi_R = 0 \,, \\ i\gamma^\mu \partial_\mu \chi_R - \gamma_5 D_z \psi_L = 0 \,. \end{cases} \tag{2.31}$$

Apply the operator $i\gamma^\mu \partial_\mu$ to the left side of eq.(2.31):

$$\begin{aligned} -\partial_\mu \partial^\mu \psi_L + \gamma_5 D_z (i\gamma^\mu \partial_\mu \chi_R) = 0 \\ -\partial_\mu \partial^\mu \chi_R + \gamma_5 D_{\bar{z}} (i\gamma^\mu \partial_\mu \psi_L) = 0 \,. \end{aligned} \tag{2.32}$$

Reapplying eq.(2.31), one finally obtains

$$\begin{aligned} -\partial_\mu \partial^\mu \psi_L + D_z D_{\bar{z}} \psi_L &= 0 \\ -\partial_\mu \partial^\mu \chi_R + D_z D_{\bar{z}} + [D_z, D_{\bar{z}}]\, \chi_R &= 0 \,. \end{aligned} \tag{2.33}$$

From the 4-dimensional point of view, the ψ_L and χ_R square masses are given by the eigenvalues of the following operators:

$$\begin{aligned} \hat{M}_L^2 \psi_L &= -D_z D_{\bar{z}} \psi_L = \; m_L^2 \; \psi_L \\ \hat{M}_R^2 \chi_R &= -D_z D_{\bar{z}} - [D_z, D_{\bar{z}}] \;\; \chi_R = \; m_R^2 \, \chi_R \,. \end{aligned} \tag{2.34}$$

Notice that we have obtained an asymmetry between the left and right square mass operators. In particular *if* $[D_z, D_{\bar{z}}] \neq 0$, *we cannot have at the same time zero modes for both chiralities* ψ_L *and* χ_R. Therefore, a gauge background with a field strength different from zero induces chirality in 4 dimensions. On the contrary, a constant background (*i.e.* a zero field strength background) cannot generate chirality.

In the case of a magnetic background with constant field strength as in eq.(2.16), it results

$$[D_z, D_{\bar{z}}] = \frac{2\pi m}{l_1 l_2} . \tag{2.35}$$

The magnitude of the non-degeneracy between left and right components depends on m and on the value of the torus area.

It is possible to rewrite this result in terms of eigenstates of the one-dimensional harmonic oscillators. Define the following operators

$$\left\{ \begin{array}{r} a \equiv \sqrt{\frac{l_1 l_2}{4\pi m}} D_{\bar{z}} \\ a^\dagger \equiv -\sqrt{\frac{l_1 l_2}{4\pi m}} D_z \, , \end{array} \right. \tag{2.36}$$

which satisfy the commutation relation

$$\left[a, a^+\right] = 1 \, . \tag{2.37}$$

In terms of a and $a^\dagger$, the squared mass operators can be rewritten as

$$\left\{ \begin{array}{rcl} \hat{M}_L^2 = \frac{2\pi m}{l_1 l_2} \, a^\dagger a & = & \frac{2\pi m}{l_1 l_2} \, \hat{j} \, , \\ \\ \hat{M}_R^2 = \frac{2\pi m}{l_1 l_2} \, (a^\dagger a + 1) & = & \frac{2\pi m}{l_1 l_2} \, (\hat{j} + 1) \, , \end{array} \right. \tag{2.38}$$

where $\hat{j}$ is the usual number operator. The eigenvalues m_L^2 and m_R^2 of the square mass operator $\hat{M}_L^2$ and $\hat{M}_R^2$ of eq.(2.34) are given by

$$\left\{ \begin{array}{r} m_L^2 = \frac{2\pi m}{l_1 l_2} \, j \\ \\ m_R^2 = \frac{2\pi m}{l_1 l_2} \, (j + 1) \, , \end{array} \right. \tag{2.39}$$

with $j = 0, 1, .., \infty$.

With the implicit choice made here, $m > 0$, it follows that only left-handed fields admit zero modes. If instead $m \to -m$, the operator a and $a^\dagger$ exchange their roles and only the right-handed fields may have a zero mode.

2.2.1 Wave functions

Consider a 6-dimensional Weyl fermion which interacts with a magnetic background, as in the previous discussions. The wave function for a fermion living on the complete $\mathcal{M}_4 \times \mathcal{T}^2$ multidimensional space can be written as

$$\Psi(x, \vec{y}) = \left(\begin{array}{c} \sum_j \psi^{(j)}(x) \, \zeta_j(\vec{y}) \\ \sum_j \chi^{(j)}(x) \, \zeta_j(\vec{y}) \end{array} \right) , \tag{2.40}$$

where the $D = 6$ wave function is expressed in terms of the eigenfunctions ζ_j of the number operator $\hat{j}$ defined in eq.(2.38), with coefficients $\psi^{(j)}$ and $\chi^{(j)}$ depending only on 4-dimensional coordinates. In order to obtain the explicit form of ζ_j, the following eigenvalue problem has to be solved,

$$a^\dagger a \, \zeta_j(\vec{y}) \quad = \quad j \, \zeta_j(\vec{y}) \, , \tag{2.41}$$

with periodicity conditions given by[2]

$$\zeta_j(\vec{y} + l_1) \quad = \quad e^{i \frac{\pi m}{l_2} y_2} \, \zeta_j(\vec{y}) \, , \tag{2.42}$$

$$\zeta_j(\vec{y} + l_2) \quad = \quad e^{-i \frac{\pi m}{l_1} y_1} \, \zeta_j(\vec{y}) \, . \tag{2.43}$$

As for any harmonic oscillator, the solutions can be obtained from that for the lower state,

$$a \zeta_{j=0}(\vec{y}) = 0 \tag{2.44}$$

applying recursively the creation operator:

$$\zeta_{j=r+1}(\vec{y}) = \sqrt{\frac{1}{r+1}} \, a^\dagger \, \zeta_{j=r}(\vec{y}) = -\sqrt{\frac{1}{r+1}} \sqrt{\frac{l_1 l_2}{4\pi m}} D_z \zeta_{j=r}(\vec{y}) \, . \tag{2.45}$$

We propose for $\zeta_{j=0}(\vec{y})$ an ansatz compatible with the periodicity condition along y_1 in eq.(2.42):

$$\zeta_{j=0}(y_1, y_2) = \sum_{n=-\infty}^{\infty} c_n(y_2) e^{i \frac{\pi m}{l_1 l_2} y_1 y_2} e^{2\pi i n \frac{y_1}{l_1}} \, . \tag{2.46}$$

Imposing the periodicity condition along y_2, eq.(2.43), into eq.(2.46), the following relation between coefficients results:

$$c_n(y_2 + l_2) = c_{n+m}(y_2) \, . \tag{2.47}$$

In order to derive the equation which defines $c_n(y_2)$, substitute

$$\begin{aligned} \partial_1 \zeta_{j=0} \quad &= \quad \sum_{n=-\infty}^{\infty} e^{i \frac{\pi m}{l_1 l_2} y_1 y_2} e^{2\pi i n \frac{y_1}{l_1}} \left(i \frac{\pi m}{l_1 l_2} y_2 + i \frac{2\pi n}{l1} \right) c_n(y_2) \\ i\partial_2 \zeta_{j=0} \quad &= \quad \sum_{n=-\infty}^{\infty} e^{i \frac{\pi m}{l1 l2} y_1 y_2} e^{2\pi i n \frac{y_1}{l_1}} \left(i\partial_2 - \frac{\pi m}{l_1 l_2} y_1 \right) c_n(y_2) \end{aligned} \tag{2.48}$$

[2]For simplicity, here we set to one the fermionic $U(1)$ charge.

into the equations of motion, eq.(2.44). The coefficients $c_n(y_2)$ are then given by

$$\partial_2 c_n(y_2) = \left(-\frac{2\pi m}{l_1 l_2} y_2 - \frac{2\pi n}{l_1} \right) c_n(y_2)\,. \tag{2.49}$$

A possible solution is of the form

$$c_n(y_2) = A_n e^{-\frac{\pi m}{l_1 l_2} y_2^2 - \frac{2\pi n}{l_1} y_2}\,, \tag{2.50}$$

The coefficient A_n is fixed by imposing the condition in eq.(2.47), which indicates that

$$A_{n+m} = A_n e^{-\pi \frac{l_2}{l_1}(2n+m)}\,, \tag{2.51}$$

implying

$$A_n = b_n e^{-\pi \frac{l_2}{l_1} \frac{n^2}{m}}\,, \tag{2.52}$$

with b_n a constant which satisfies $b_{n+m} = b_n$. Therefore it exists $|m|$ arbitrary constant coefficients. This result implies that we have $|m|$ independent solutions for the zero mode characterized by the integer number $\rho = 0, ..., m-1$. It is, therefore, possible to re-parametrize the index n as $n \to nm + \rho$. The m replicas of the zero mode are the representants of any subspace $\mathcal{H}_{m,n}$ in which the symmetry $\mathbb{Z}_m$ associated to the operators $K^{1,2}$ splits the Hilbert space $\mathcal{H}_m$, as discussed previously. Summarizing, the form of the lightest wave function is

$$\zeta_{j=0}(y_1, y_2) \;=\; \sum_{\rho=0}^{m-1} b_\rho\, \zeta_{j=0,\rho}(y_1, y_2)\,, \tag{2.53}$$

where b_ρ are arbitrary coefficients and

$$\zeta_{j=0,\rho}(y_1, y_2) = \mathcal{N} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi m}{l_1 l_2}(y_2 + n l_2 + \frac{\rho l_2}{m})^2} e^{2\pi i (mn+\rho)\frac{y_1}{l_1}} e^{i \frac{\pi m}{l_1 l_2} y_1 y_2}\,. \tag{2.54}$$

The normalization $\mathcal{N}$ of the independent wave functions can be calculated from

$$1 = \int_{T_2} \left(\zeta_{(j=0,\rho)}\right)^\dagger\, \zeta_{(j=0,\rho)}\,, \tag{2.55}$$

resulting in

$$\mathcal{N} = \sqrt[4]{2\frac{l_2}{l_1}} \sqrt{\frac{1}{l_1 l_2}}\,. \tag{2.56}$$

Finally, the zero mode wave function reads

$$\zeta_{j=0,\rho} \quad = \quad \sqrt[4]{2\frac{l_2}{l_1}}\sqrt{\frac{1}{l_1 l_2}}\sum_{n=-\infty}^{\infty} e^{-\frac{\pi m}{l_1 l_2}(y_2+nl_2+\frac{\rho l_2}{m})^2} e^{2\pi i(mn+\rho)\frac{y_1}{l_1}} e^{i\frac{\pi m}{l_1 l_2}y_1 y_2} \, . \tag{2.57}$$

Notice that for the case $m > 1$, the different independent solutions are localized in different points of the extra dimensions. We have represented this fact in fig. 2.1, fig. 2.2 and fig. 2.3. In particular it shows that for $m = 3$, starting from only one $D = 6$ Weyl fermion, it is possible to obtain three $D = 4$ chiral fermions localized in different points of the extra dimensions [77]. This could be hypothesized to have something to do with the origin of the three fermion generations in nature.

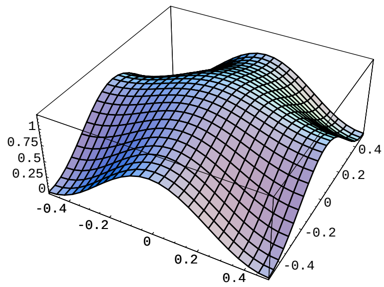

Figure 2.1: Gaussian behaviour of $|\zeta_{j=0,\rho=0}|^2$ in the case of $m = 1$

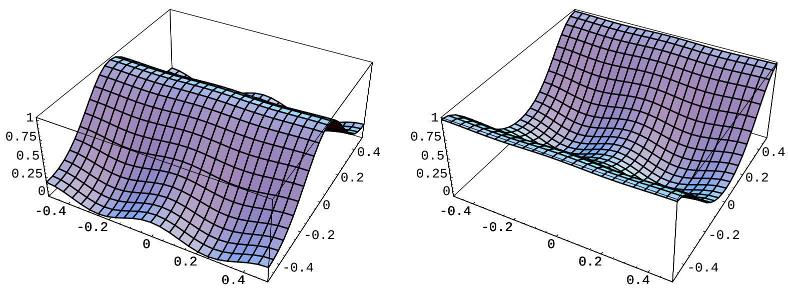

Figure 2.2: Behaviour of $|\zeta_{j=0,\rho=0,1}|^2$, respectively, in the case of $m = 2$: the two independent solutions $\rho = 0, 1$ are localized in two different points of the torus.

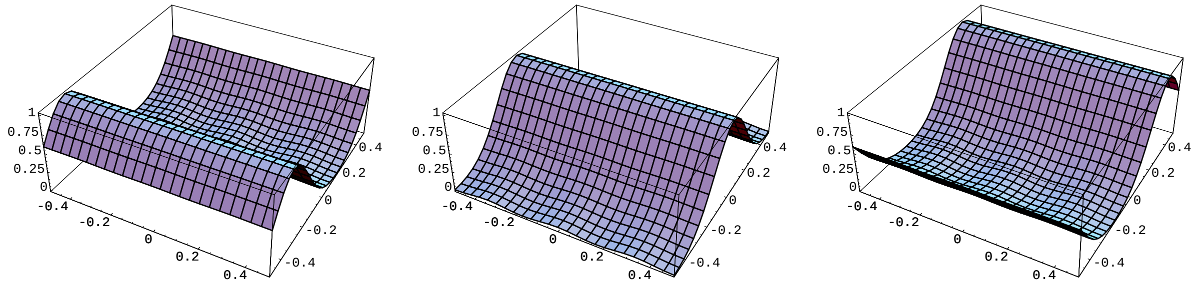

Figure 2.3: Behaviour of $|\zeta_{j=0,\rho=0,1,2}|^2$ in the case of $m = 3$: the three independent solutions $\rho = 0, 1, 2$ are localized in three different points of the torus.

Let us conclude this chapter with the expression for the heavier modes. We obtain the following expression wave for the j-th mode wave function:

$$\zeta_{j,\rho} = \sqrt[4]{2\frac{l_2}{l_1}}\sqrt{\frac{1}{l_1 l_2}}e^{i\frac{\pi m}{l_1 l_2}y_1 y_2}\sum_{n=-\infty}^{\infty} H_j(y_2 + nl_2 + \frac{\rho l_2}{m})e^{-\frac{\pi m}{l_1 l_2}(y_2+nl_2+\frac{\rho l_2}{m})^2}e^{2\pi i(mn+\rho)\frac{y_1}{l_1}}\,, \tag{2.58}$$

with $H_j(y_2 + nl_2 + \frac{\rho l_2}{m})$ being the Hermite polynomials.

Chapter 3

Phenomenology of symmetry breaking from extra dimensions

Motivated by the electroweak hierarchy problem, we consider theories with two extra dimensions in which the four-dimensional scalar fields are components of gauge boson in full space. We explore the Nielsen-Olesen instability for $SU(N)$ on a torus, in the presence of a magnetic background. A field theory approach is developed, computing explicitly the minimum of the complete effective potential, including tri-linear and quartic couplings and determining the symmetries of the stable vacua. We also develop appropriate gauge-fixing terms when both Kaluza-Klein and Landau levels are present and interacting, discussing the interplay between the possible six and four dimensional choices. The equivalence between coordinate dependent and constant Scherk-Schwarz boundary conditions -associated to either continuous or discrete Wilson lines- is analyzed.

In Section 3.1, general theoretical arguments prove the existence of absolute minima, for $SU(N)$. Boundary conditions depending on the extra coordinates are shown to be equivalent to constant ones and the expected symmetry breaking patterns for the stable vacua are determined. In Section 3.2 the problem is reformulated in terms of the $6D$ $SU(N)$ Lagrangian. Next we obtain the complete effective four-dimensional Lagrangian out of the explicit integration of the $6D$ Lagrangian over the torus surface, for the $SU(2)$ case; appropriate gauge-fixing conditions are proposed and developed in detail as well. In Section 3.3 the stable minima of the complete four-dimensional potential and the resulting physical spectra is identified, for the $SU(2)$ case. The last step of this procedure is done numerically and the results are then compared with the symmetry breaking patterns expected from the general theoretical analysis developed in Section 3.1. In Section 3.4 we conclude. The Appendices A and B contain supplementary arguments and develop further technical tools.

3.1 Vacuum energy

Consider a $6D$ $SU(N)$ gauge theory, with generators λ^a defined by $\mathrm{Tr}[\lambda^a\lambda^b] = \delta^{ab}/2$ and $[\lambda^a, \lambda^b] = if^{abc}\lambda^c$. The Yang Mills Lagrangian reads

$$\mathcal{L}_6 = -\frac{1}{2}\,\mathrm{Tr}[\mathbf{F}_{MN}\mathbf{F}^{MN}] = -\frac{1}{4}\,\mathbf{F}^a_{MN}\mathbf{F}_a^{MN}\;, \tag{3.1}$$

where

$$\mathbf{F}^a_{MN} \;\;=\;\; \partial_M\mathbf{A}^a_N - \partial_N\mathbf{A}^a_M + gf^{abc}\mathbf{A}^b_M\mathbf{A}^c_N\;, \tag{3.2}$$

and $\mathbf{A}^a_M$ are the gauge fields in the adjoint representation of the group. Throughout the chapter, Greek (Latin) indices will denote the ordinary (extra) coordinates. The two extra dimensions are compactified on an orthogonal torus $\mathcal{T}^2$, with compactification lengths l_1, l_2, and area $\mathcal{A} = l_1 l_2$. In what follows, we will denote by x the four Minkowski coordinates and by y the two extra coordinates.

We assume a constant field strength pointing to an arbitrary direction in gauge space. We also assume $4D$ Poincaré invariance. In accordance with it, the background can only be of the form $B_M = (0, B^a_i(y))$. The gauge fields can then be parametrized in terms of that classical background, B^a_M, and the fluctuations A^a_M,

$$\mathbf{A}^a_M(x,y) = B^a_M(y) + A^a_M(x,y)\;, \tag{3.3}$$

allowing to decompose the total field strength as

$$\mathbf{F}^a_{MN}(x,y) = G^a_{MN} \;+\; F^a_{MN}(x,y)\,, \tag{3.4}$$

with G_{MN} given by

$$G^a_{\mu\nu} = 0\,, \qquad G^a_{\mu i} = 0\,, \qquad G^a_{ij} = \partial_i B^a_j - \partial_i B^a_j + gf^{abc}B^b_i B^c_j\,. \tag{3.5}$$

In what follows, $B_i(y)$ and G_{ij} will be denoted *imposed* background and field strength, respectively, which do not necessarily coincide with those of a true -stable- vacuum configuration. The latter will be instead dubbed *total*.

To live on a torus implies to specify boundary conditions, which describe how fields transform under translations by l_1 and l_2. Let T_i be the embedding of such translations in gauge space. Upon their action, gauge fields in the adjoint representation can vary at most by a gauge transformation,

$$\mathbf{A}_M(x, y + l_i) = T_i(y)\,\mathbf{A}_M(x,y)\,T_i^\dagger(y) + \frac{i}{g}\,T_i(y)\,\partial_M T_i^\dagger(y)\,. \tag{3.6}$$

Translations T_i must, in general, commute up to an element of the center of the group,

$$T_2^{-1}(y_1,y_2)\,T_1^{-1}(y_1,y_2+l_2)\,T_2(y_1+l_1,y_2)\,T_1(y_1,y_2) = e^{2\pi i(k+\frac{m}{N})}\,, \tag{3.7}$$

where k and m are integers, with m being the *'t Hooft non-abelian flux* [68], a gauge invariant quantity constrained to take values between 0 and $(N-1)$.

Given a set of T_i, the possible backgrounds B_i are constrained by Eq. (3.6), implying

$$
\begin{aligned}
A_M(x, y+l_i) &= T_i(y)\, A_M(x,y)\, T_i^\dagger(y)\,, && (3.8)\\
F_{MN}(x, y+l_i) &= T_i(y)\, F_{MN}(x,y)\, T_i^\dagger(y)\,, && (3.9)\\
B_j(y+l_i) &= T_i(y)\, B_j(y)\, T_i^\dagger(y) + \frac{i}{g}\, T_i(y)\, \partial_j T_i^\dagger(y)\,, && (3.10)\\
G_{MN} &= T_i(y)\, G_{MN}\, T_i^\dagger(y)\,. && (3.11)
\end{aligned}
$$

Instability

For a $SU(N)$ theory on a two-dimensional torus, an expansion around a constant field strength corresponds to a background configuration that satisfies the equations of motion, but it is not stable. A simple argument goes as follows. Given a constant G_{12}, the only mass term present in the $6D$ Lagrangian for the $6D$ field excitations is

$$
-g f^{abc} A_1^b A_2^c\, G_a^{12}\,. \tag{3.12}
$$

Because the background field strength G_{12} is a non-zero Lorentz constant, the anticommutativity of f^{abc} implies then the presence in the Lagrangian of a field with negative mass, as can be seen rewriting Eq. (3.12) in the diagonal basis[1]. In other words, the mass matrix defined by Eq. (3.12) is a traceless quantity and, for $G_{12} \neq 0$, it necessarily has at least one positive and one negative mass eigenvalue[2].

The instability argument for a background with constant field strength can be also discussed from a $4D$ point of view. The $4D$ Lagrangian is

$$
\begin{aligned}
\mathcal{L}_4 &= \int_{\mathcal{T}^2} d^2y\, \mathcal{L}_6 = -\frac{1}{2}\int_{\mathcal{T}^2} d^2y\, \mathrm{Tr}\left[\mathbf{F}_{MN}\mathbf{F}^{MN}\right] = \\
&= -\frac{1}{2}\int_{\mathcal{T}^2} d^2y\, \mathrm{Tr}\left[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} + 2\mathbf{F}_{\mu i}\mathbf{F}^{\mu i} + \mathbf{F}_{ij}\mathbf{F}^{ij}\right]. && (3.13)
\end{aligned}
$$

Our aim is to identify the possible degenerate vacuum solutions consistent with $\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} = 0$ and compatible with the boundary conditions. $4D$ Lorentz and $4D$ translation invariance on a flat $\mathcal{M}_4 \times \mathcal{T}^2$ manifold also require that, at the minimum, $\mathbf{F}^{\mu i} = 0$. The third term in Eq. (3.13) is positive semi-definite,

$$
\int_{\mathcal{T}^2} d^2y\, \mathrm{Tr}\left[\mathbf{F}_{ij}^2\right] \geq 0\,. \tag{3.14}
$$

[1] Other possible mass terms, resulting after fixing the gauge for the excitation fields, only produce symmetric terms, which cannot cancel the antisymmetric contributions in Eq. (3.12).

[2] This is unlike the $U(N)$ case, for instance, where the $U(1)$ part is not subject to such a constraint.

For a $SU(N)$ gauge theory on a $2D$ torus, the energy is not bounded from below by any topological quantity[3]. Consequently, the absolute minimum should correspond to the lower limit of the inequality Eq. (3.14), implying

$$\mathbf{F}^a_{ij}\Big|_{min} \equiv \widetilde{G}^a_{ij} \,=\, 0, \quad \forall\, i,\, j,\, a \qquad \Rightarrow \qquad \widetilde{G}^a_{ij} = G^a_{ij} \,+\, F^a_{ij}\Big|_{min} = 0\,, \tag{3.15}$$

where Eq. (3.4) has been used. In the above and from now on we denote with $\sim$ the quantities pertaining to the *total* stable vacua, which has vanishing field strength, $\widetilde{G}^a_{ij} = 0$.

In other words, the original *imposed* configuration, with constant background field strength, G^a_{ij}, is not stable. In order to satisfy Eq. (3.15) the scalars contained in the $4D$ potential,

$$V = \frac{1}{2}\int_{T^2} d^2y\, \mathrm{Tr}[F^2_{ij} + 2\, G_{ij}\, F_{ij}]\,, \tag{3.16}$$

will have to develop vacuum expectation values, allowing the system to evolve towards a stable vacuum. That is, it is to be expected that the system will respond to the *imposed* background through a pattern alike to that of $4D$ spontaneous symmetry breaking.

Furthermore, as the total vacuum energy will correspond to

$$E_{tot} = \frac{1}{2}\int d^4x \int_{T^2} d^2y\ \mathrm{Tr}[\mathbf{F}^2_{ij}\big|_{min}] = 0\,, \tag{3.17}$$

the absolute minima will have to be reached from the initial *imposed* background through a pattern of scalar vacuum expectation values which, at the classical level, **do not** contribute to the cosmological constant, which thus remains being zero.

The true vacuum

The true vacuum should correspond to a configuration of zero energy, $\widetilde{G}_{MN} = 0$, as explained above. Let $\widetilde{B}_i(y)$ be such a stable background configuration, whose precise form remains to be found. $\widetilde{B}_i(y)$ can be interpreted as the sum of the *imposed* background $B_i(y)$ plus that resulting from the system response. A $SU(N)$ gauge configuration of zero energy is a pure gauge and may be expressed by

$$\widetilde{B}_i(y) = \frac{i}{g} U(y)\partial_i U^\dagger(y)\,, \tag{3.18}$$

where U is a $SU(N)$ gauge transformation. The problem of finding the non-trivial vacuum of the theory reduces, then, to build a $SU(N)$ gauge transformation $U(y)$ compatible with

[3]Notice the difference between $SU(N)$ and $U(N)$ on T_2. In $U(N)$, $\int_{T^2} \mathrm{Tr}\,[\mathbf{F}^2_{ij}] \geq (1/4)\int_{T^2} |\mathrm{Tr}\,(\epsilon_{\mu\nu}F^{\mu\nu})|^2$, which may be non-zero.

the boundary conditions. Substituting Eq. (3.18) into Eq. (3.6), it follows that U must satisfy

$$U(y+l_i) = T_i(y)\,U(y)\,V_i^\dagger\,, \tag{3.19}$$

where V_i are arbitrary constant elements of $SU(N)$, only subject to the constraint

$$V_1^{-1}\,V_2^{-1}\,V_1\,V_2 \;=\; e^{2\pi i(k+\frac{m}{N})}\,. \tag{3.20}$$

For $SU(N)$ on a $2D$ torus, it is always possible [81–83] to solve recursively the boundary conditions (3.19) and consequently such an U exists.

Under a gauge transformation $S \in SU(N)$, the embeddings of translations transform as

$$T_i'(y) = S(y+l_i)\,T_i(y)S^\dagger(y)\,. \tag{3.21}$$

In order to catalogue the possible degenerate vacua, it is useful to work in a gauge that we will denote as $6D$-*background symmetric gauge*: that in which the *total* vacuum gauge configuration is trivial, $\widetilde{B}_M^{sym} = 0$. Upon the gauge transformation $S = U^\dagger$, with U defined in Eq. (3.19), it results

$$T_i^{sym} = U^\dagger(y+l_i)T_i(y)U(y) = V_i\,, \qquad \widetilde{B}_M^{sym} = 0\,. \tag{3.22}$$

In this gauge the background is then zero and the constant matrices V_i coincide with the boundary conditions. To classify the classical degenerate minima is then tantamount to classify the possible constant matrices V_i. The symmetries of the vacuum correspond to those generators commuting with all V_i. V_i can be parametrized as

$$V_i \equiv e^{2\pi i\alpha_i^a\lambda^a}\,, \tag{3.23}$$

with α_i^a being arbitrary constants only subject to the consistency condition (3.20). Two main cases can occur depending on whether the value of m in Eq. (3.7) is equal to zero or not. Notice that:

- For $m = 0$, as the embeddings of translations V_i commute, it is possible to perform a non-periodic gauge transformation leading to gauge fields which transform "periodically", while the boundary conditions are reabsorbed in the vacuum expectation values of scalar fields (Hosotani mechanism).

- For $m = 1$, on the contrary, as the V_i do not commute, such a transformation to periodic boundary conditions is not achievable.

3.1.1 Trivial 't Hooft flux: $m = 0$

The name reminds that, in this case, the embedding of translations in gauge space commute and all classical vacuum solutions are degenerate in energy with the trivial vacuum, which is $SU(N)$ symmetric.

For $m = 0$, the V_i constant matrices commute, constraining the possible λ^a in Eq. (3.23) to belong to the $(N-1)$ generators of the Cartan subalgebra. The vacua are thus characterized by $2(N-1)$ real continuous parameters α_i^a, $0 \leq \alpha_i^a < 1$. These α_a^i are non-integrable phases, which only arise in a topologically non-trivial space and cannot be gauged-away. Their values must be dynamically determined at the quantum level: only at this level the degeneracy among the infinity of classical vacua is removed [60–62].

The solution with $\alpha_i^a = 0$ is the trivial, $SU(N)$ symmetric, one. For non-zero α_i^a values, the residual gauge symmetries are those associated with the generators that commute with V_i. As V_1 and V_2 commute, the rank of $SU(N)$ cannot be lowered [88] and thus the maximal symmetry breaking pattern that can be achieved is

$$SU(N) \longrightarrow U(1)^{N-1}. \tag{3.24}$$

The spectrum of the $4D$ fields corresponding to the Cartan subalgebra is that of an ordinary Kaluza-Klein (KK) tower,

$$M^2_{n_1,n_2} = 4\pi^2 \left[\frac{n_1^2}{l_1^2} + \frac{n_2^2}{l_2^2} \right] , \qquad n_1, n_2 \in \mathbb{Z} , \tag{3.25}$$

whereas for the rest of the fields, that is, fields corresponding to generators that do not commute with all V_i, the spectrum is expected to be of the form

$$M^2_{n_1,n_2} = 4\pi^2 \left[\frac{(n_1 + \sum_{a=1}^{N-1} q^a \alpha_1^a/2\,)^2}{l_1^2} + \frac{(n_2 + \sum_{a=1}^{N-1} q^a \alpha_2^a/2\,)^2}{l_2^2} \right] , \tag{3.26}$$

where q^a are the field charges, expressed in units of the charge of the fundamental representation. These type of spectra are characteristic of Scherk-Schwarz symmetry breaking scenarios [58–62, 95, 97].

In the simplest case of $SU(2)$, that will be of interest for us in the following sections, the two V_i matrices may be chosen[4] to be for instance $V_1 = e^{\pi i \alpha_1 \sigma_3}$ and $V_2 = e^{\pi i \alpha_2 \sigma_3}$.

The mass spectrum for the fields A_M^3 coincides with the KK spectrum (3.25), whereas for fields which do not belong to the Cartan subalgebra is is given by

$$M^2_{n_1,n_2} = 4\pi^2 \left[\frac{(n_1 \pm \alpha_1\,)^2}{l_1^2} + \frac{(n_2 \pm \alpha_2\,)^2}{l_2^2} \right] , \tag{3.27}$$

[4]The direction $a = 3$ is only a possible choice; obviously the choice of gauge direction in the parametrization is arbitrary. It bears no relationship with the gauge direction chosen for the *imposed* background.

as $q^a = 2$ for fields in the adjoint representation. There are no massless modes in this sector, for non-zero α_i. The expected symmetry breaking pattern is thus

$$SU(2) \longrightarrow U(1)\,. \tag{3.28}$$

3.1.2 Non-trivial 't Hooft flux: $m \neq 0$

In this case, all solutions exhibit symmetry breaking, even at the classical level. The embeddings of translations in gauge space do not commute, Eq. (3.7), and the same holds then for the constant matrices V_i [68, 98–101]. In consequence, the symmetry breaking pattern lowers the rank of the group [102].

Furthermore, the consistency condition in Eq. (3.7), entails now the quantization of the α_i parameters defining V_i. Indeed, it is always possible to choose such V_i of the form [70, 103]:

$$\begin{cases} V_1 &= P^{s_1}\, Q^{t_1} \\ V_2 &= P^{s_2}\, Q^{t_2} \end{cases}, \tag{3.29}$$

where $P \equiv e^{i\pi(N-1)/N}\, diag(1, e^{2\pi i \frac{1}{N}}, ..., e^{2\pi i \frac{N-1}{N}})$, $Q_{ij} \equiv e^{i\pi(N-1)/N}\, \delta_{ij-1}$, satisfying $P^N = Q^N = e^{i\pi(N-1)}$ and $P\,Q = e^{2\pi i/N}\, Q\, P$. The parameters s_i, t_i are integers that assume values between 0 and $N-1$ and that have to satisfy the consistency condition

$$s_1\, t_2 \;-\; s_2\, t_1 \;=\; m\,. \tag{3.30}$$

Consider for instance the first choice in Eq. (3.29). It follows that

$$\begin{cases} V_1^N &= e^{i\pi(s_1+t_1)(N-1)}\, 1\!\!1 \\ V_2^N &= e^{i\pi(s_2+t_2)(N-1)}\, 1\!\!1\,, \end{cases} \tag{3.31}$$

implying that the non-integrable phases in Eq. (3.23) are not free parameters, but quantized ones even at the classical level. Let's define $\mathcal{K}_1 = g.c.d.\,(m, N)$ and $\mathcal{K}_2 = g.c.d.(s_1,\, s_2,\, t_1,\, t_2, N)$. Using Eq.(3.30), it is possible to prove that $\mathcal{K}_2 \leq \mathcal{K}_1$ and that $\mathcal{K}_1/\mathcal{K}_2 \in \mathbf{Z}$. In terms of these two parameters, the residual symmetry group has dimension $(\mathcal{K}_1\, \mathcal{K}_2 - 1)$, consistent with the following gauge symmetry breaking pattern [83, 104]:

$$SU(N) \to SU(\mathcal{K}_2)^{\frac{\mathcal{K}_1}{\mathcal{K}_2}} \times U(1)^{\frac{\mathcal{K}_1}{\mathcal{K}_2}-1}\,. \tag{3.32}$$

For $\mathcal{K}_1 = 1$ (which implies $\mathcal{K}_2 = 1$), $SU(N)$ is thus completely broken.

It can be shown that the mass spectrum is arranged along towers of fields [83, 104] whose masses can be expressed as

$$(M^a_{n_1,n_2})^2 = 4\pi^2 \left[\frac{(n_1 + \beta_1^a/N)^2}{l_1^2} + \frac{(n_2 + \beta_2^a/N)^2}{l_2^2}\right], \tag{3.33}$$

with quantized parameters β_i^a, as a consequence of Eq. (3.31), $\beta_i^a = 0,...,N-1$. Some gauge fields can thus be massless: for $\mathcal{K}_1 > 1$, there are $(\mathcal{K}_1\mathcal{K}_2 - 1)$ massless modes; otherwise, if $\mathcal{K}_1 = 1$ both β_i^a cannot be simultaneously zero and no massless modes remain. In summary, these type of spectra are characteristic of constant discrete Scherk-Schwarz boundary condition scenarios [105]: they are alike to the Scherk-Schwarz patterns obtained for $m = 0$, albeit with the parameters β_i quantized.

As an illustration, let us particularize again to the $SU(2)$ case. The only possible non-zero value of m is then $m = 1$, for which a possible choice for the P and Q matrices is $P = i\sigma_3$ and $Q = i\sigma_1$, with V_i given by

$$\left\{ \begin{array}{l} V_1 = i\sigma_3 \\ V_2 = i\sigma_1 \end{array} \right. \qquad \text{or} \qquad \left\{ \begin{array}{l} V_1 = i\sigma_1 \\ V_2 = i\sigma_3 \end{array} \right. . \tag{3.34}$$

As $\mathcal{K}_1 = \mathcal{K}_2 = 1$, Eq.(3.32) entails that the expected symmetry breaking pattern is

$$SU(2) \longrightarrow \varnothing \,,$$

even at the classical level. Three towers of fields result, with masses given by

$$M^2_{n_1,n_2} = \left\{ \begin{array}{l} 4\pi^2 \left[\dfrac{(n_1 + 1/2)^2}{l_1^2} + \dfrac{n_2^2}{l_2^2} \right] \\[3ex] 4\pi^2 \left[\dfrac{(n_1 + 1/2)^2}{l_1^2} + \dfrac{(n_2 + 1/2)^2}{l_2^2} \right] \\[3ex] 4\pi^2 \left[\dfrac{n_1^2}{l_1^2} + \dfrac{(n_2 + 1/2)^2}{l_2^2} \right] . \end{array} \right. \tag{3.35}$$

These expressions allow no zero modes and thus the $SU(2)$ gauge symmetry is indeed completely broken[5].

To conclude this Section, we have seen that for $SU(N)$ on a $2D$ torus, the y-dependent boundary conditions are equivalent to constant Scherk-Schwarz boundary conditions (V_i). For the case of trivial-'t Hooft flux, $m = 0$, the treatment shows them to be equivalent to boundary conditions associated to continuous Wilson lines, while for $m \neq 0$ they are equivalent to boundary conditions associated to discrete Wilson lines.

[5]With the particular choice in Eq. (3.34) the three towers in Eq. (3.35) would correspond to the gauge directions $a = 1, 2, 3$, respectively.

3.2 The effective Lagrangian theory

In the rest of the chapter, we will analyze the pattern of symmetry breaking within a completely different approach: the identification of the minimum of the effective $4D$ potential, after integrating the initial $6D$ Lagrangian -with a constant background field strength- over the extra dimensions. To find and verify explicitly the form of the true vacuum, solving the Nielsen-Olesen instability on the torus, we will obtain the $4D$ scalar potential and minimize it. After some general considerations for $SU(N)$, we will treat in full detail the $SU(2)$ case and compare the resulting spectra with those predicted in the previous Section.

3.2.1 The 6-dimensional $SU(N)$ Lagrangian

The Yang-Mills Lagrangian Eq. (3.1) can be rewritten in terms of the *imposed* background and its fluctuations as

$$\mathcal{L}_{YM} = -\frac{1}{4}(G^a_{MN} + F^a_{MN})^2 = \mathcal{L}^{(0)}_A + \mathcal{L}^{(1)}_A + \mathcal{L}^{(2)}_A + \mathcal{L}^{(3)}_A + \mathcal{L}^{(4)}_A \,, \qquad (3.36)$$

where the Lagrangian terms corresponding to $i = 0, 1, 2, 3, 4$ fluctuation fields are, explicitly,

$$\mathcal{L}^{(0)}_A = -\frac{1}{4}\, G^a_{MN} G^{MN}_a \qquad (3.37)$$

$$\mathcal{L}^{(1)}_A = -\frac{1}{2}\, G^a_{MN}(D^M A^{N\,a} - D^N A^{M\,a})] \qquad (3.38)$$

$$\mathcal{L}^{(2)}_A = -\frac{1}{2}[D_M A^a_N\, D^M A^{N\,a} - D_M A^a_N\, D^N A^{M\,a} + g f^{abc} G^a_{MN} A^M_b A^N_c] \qquad (3.39)$$

$$\mathcal{L}^{(3)}_A = -\frac{1}{2}\, g f^{abc}(D^M A^{N\,a} - D^N A^{M\,a}) A^b_M A^c_N \qquad (3.40)$$

$$\mathcal{L}^{(4)}_A = -\frac{1}{4}\, g^2 f^{abc} f^{amn} A^b_M A^c_N A^M_m A^N_n \,. \qquad (3.41)$$

The form of G_{MN} was given in Eq.(3.5), while

$$F^a_{MN} = D_M A^a_N - D_N A^a_M + g f^{abc} A^b_M A^c_N, \qquad (3.42)$$

with D_M being the *imposed*-background covariant derivative,

$$D_M A^a_N \equiv \partial_M A^a_N - g f^{abc} A^b_N B^c_M \,, \qquad (3.43)$$

satisfying

$$[D_M, D_N] = -i\, g\, G_{MN} \,. \qquad (3.44)$$

Notice that classically $\mathcal{L}_A^{(1)} = 0$, as the *imposed* background satisfies the stationarity condition given by the equations of motion, $D^a{}_M G^{MN} = 0$, although we will see below this it is not a stable vacuum configuration.

A possible choice for the *imposed* background, compatible with constant G_{MN}, is

$$B_i(y) = -\epsilon_{ij}\,\frac{2\pi}{g}\left(k+\frac{m}{N}\right)\frac{y_j}{\mathcal{A}}\,\hat{\lambda}\,, \tag{3.45}$$

where $\hat{\lambda}$ denotes an arbitrary direction in gauge space, leading to

$$G_{12} = \frac{4\pi(k+\frac{m}{N})}{g\,\mathcal{A}}\,\hat{\lambda} \equiv \frac{2}{g}\,\mathcal{H}\,\hat{\lambda}\,. \tag{3.46}$$

The quantity $\mathcal{H}$ so defined can be interpreted as a quantized abelian magnetic flux over the torus surface (up to some factors):

$$\frac{1}{\mathcal{A}}\int_{T^2} d^2y\;(\partial_1 B_2 - \partial_2 B_1) = \frac{1}{\mathcal{A}}\int_{T^2} d^2y\,G_{12} = \frac{2}{g}\,\mathcal{H}\,\hat{\lambda}\,. \tag{3.47}$$

The above choice for B_i is consistent with the following embeddings of translations:

$$T_i(y) = e^{\epsilon_{ij}\pi i(k+\frac{m}{N})\frac{y_j}{l_j}\,\hat{\lambda}}\,, \tag{3.48}$$

which satisfy the conditions in Eq. (3.7), when $\hat{\lambda}$ is chosen as the $SU(N)$ generator of the Cartan subalgebra of the form $\hat{\lambda} = diag(1,1,\cdots,1-N)$.

The boundary conditions for the fluctuation fields can be most conveniently expressed choosing the bases in Poincaré space defined by $z(\overline{z}) \equiv (y_1 \pm i y_2)/\sqrt{2}$ and $A^a_{z(\overline{z})} \equiv (A_1^a \mp iA_2^a)/\sqrt{2}$ and in gauge space by $[\lambda_a, \hat{\lambda}] = q^a \lambda^a$. In these bases,

$$\left\{\begin{array}{lcl} A^a_M(y_1+l_1, y_2) & = & e^{i\,\pi(k+\frac{m}{N})\frac{y_2}{l_2}\,q^a}\;A^a_M(y_1,y_2) \\ A^a_M(y_1, y_2+l_2) & = & e^{-i\,\pi(k+\frac{m}{N})\frac{y_1}{l_1}\,q^a}\;A^a_M(y_1,y_2)\,, \end{array}\right. \tag{3.49}$$

$$D^a_z = \partial_z - \frac{\mathcal{H}}{2}\,\overline{z}\,q^a \quad , \quad D^a_{\overline{z}} = \partial_{\overline{z}} + \frac{\mathcal{H}}{2}\,z\,q^a \quad \text{with} \quad [D^a_z, D^a_{\overline{z}}] = \mathcal{H}\,q^a\,. \tag{3.50}$$

The non-commutativity of the *imposed*-background covariant derivatives, acting on charged fields, illustrates that translations of arbitrary length along the two extra dimensions do not commute. In order to determine the physical spectrum, all terms in the Lagrangian in Eqs.. (3.37)-(3.41) will have to be considered.

Total background

Were the Lagrangian formally expanded instead around an hypothetical *total* minimum with background $\widetilde{B}_M(y)$, Eq. (3.15), and its fluctuations[6], the corresponding $\widetilde{G}_{MN}$ would vanish,

$$\widetilde{G}_{MN} = \frac{i}{g}[\widetilde{D}_M, \widetilde{D}_N] = 0\,, \tag{3.51}$$

with $\widetilde{D}_M$ given by

$$\widetilde{D}_M A^a_N \equiv \partial_M A^a_N - g f^{abc} A^b_N \widetilde{B}^c_M\,. \tag{3.52}$$

No tachyonic mass would be present then in the Lagrangian and, to extract the physical spectrum, it would be enough to consider only terms with two fluctuation fields,

$$\widetilde{\mathcal{L}}^{(2)}_A \equiv -\frac{1}{2}[\widetilde{D}_M A^a_N\, \widetilde{D}^M A^{N\,a} - \widetilde{D}_M A^a_N\, \widetilde{D}^N A^{M\,a}]\,. \tag{3.53}$$

Below we will explicitly explore the dynamical evolution of the system from the *imposed* background $B_M(y)$ to the *total* stable one, $\widetilde{B}_M(y)$, in the $SU(2)$ case.

3.2.2 The 6-dimensional $SU(2)$ Lagrangian

We particularize now the discussion to a $SU(2)$ gauge theory, with generators $\lambda^a = \sigma^a/2$, where $a = 1, 2, 3$ and σ^a denote the Pauli matrices. The commutativity condition for the embeddings of translations in gauge space, Eq. (3.7), reduces now to the values ± 1, as m can take only two values, $m = 0, 1$, while k keeps being an arbitrary integer. A possible choice for the *imposed* background is one pointing towards the third gauge direction, i.e. $\hat{\lambda} = \sigma_3/2$, whose replacement in Eqs. (3.45-3.49), defines the background and boundary conditions for this case. The gauge indices for fields in the adjoint representation are $a = +, -, 3$, with

$$\begin{cases} \lambda^+ = \frac{1}{\sqrt{2}}(\lambda_1 + i\lambda_2) \\ \lambda^- = \frac{1}{\sqrt{2}}(\lambda_1 - i\lambda_2) \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{A}^+_M = \frac{1}{\sqrt{2}}(\mathbf{A}^1_M - i\mathbf{A}^2_M) \\ \mathbf{A}^-_M = \frac{1}{\sqrt{2}}(\mathbf{A}^1_M + i\mathbf{A}^2_M) \end{cases}, \tag{3.54}$$

where $M = \mu, z, \overline{z}$. For those fields, the charges with respect to the *imposed* background are $q^a = +2, -2, 0$, in units of the charge of the fundamental representation, $q_f = 1/2$.

Consider the various components of the Yang-Mills Lagrangian, Eqs. (3.37)-(3.41), for the particular case of $SU(2)$. Working in the basis of Eq.(3.54), the Lagrangian without gauge fixing terms can now be explicitly expanded as

$$\mathcal{L}_{6D} = \mathcal{L}_{\mu\nu} + \mathcal{L}_{ij} + \mathcal{L}_{\mu i}\,, \tag{3.55}$$

[6] A^a_M is used throughout the chapter to generically denote excitations with respect to the background included in any definition of the covariant derivative.

where

$$\mathcal{L}_{\mu\nu} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a \tag{3.56}$$

$$\begin{aligned}
\mathcal{L}_{ij} &= 2\,\mathcal{H}\left(A^-_{\bar{z}}A^+_z - A^+_{\bar{z}}A^-_z\right) + \frac{1}{2}\left[(\partial_{\bar{z}}A^3_z)^2 + (\partial_z A^3_{\bar{z}})^2 - 2\,(\partial_z A^3_{\bar{z}})(\partial_{\bar{z}}A^3_z)\right] \\
&+ \left[(D_{\bar{z}}A^+_z)(D_{\bar{z}}A^-_z) + (D_zA^+_{\bar{z}})(D_zA^-_{\bar{z}}) - (D_zA^+_{\bar{z}})(D_{\bar{z}}A^-_z) - (D_{\bar{z}}A^+_z)(D_zA^-_{\bar{z}})\right] \\
&- g^2\left[\frac{1}{2}(A^+_zA^-_{\bar{z}} - A^+_{\bar{z}}A^-_z)^2 + A^3_zA^3_{\bar{z}}\left(A^+_zA^-_{\bar{z}} + A^-_zA^+_{\bar{z}}\right)\right] \\
&- g^2\left[A^3_zA^3_zA^+_{\bar{z}}A^-_{\bar{z}} + \text{h.c.}\right] + ig\left(A^+_zA^-_{\bar{z}} - A^+_{\bar{z}}A^-_z\right)\left(D_{\bar{z}}A^3_z - D_zA^3_{\bar{z}}\right) \\
&+ ig\left[\left(A^3_zA^+_{\bar{z}} - A^3_{\bar{z}}A^+_z\right)\left(D_{\bar{z}}A^-_z - D_zA^-_{\bar{z}}\right) - \text{h.c.}\right] ,
\end{aligned} \tag{3.57}$$

$$\begin{aligned}
\mathcal{L}_{\mu i} &= g^2(A^+_\mu A^\mu_-(2A^3_{\bar{z}}A^3_z + A^+_zA^-_{\bar{z}} + A^+_{\bar{z}}A^-_z) + A^3_\mu A^\mu_3(A^+_zA^-_{\bar{z}} + A^+_{\bar{z}}A^-_z) \\
&- \left[A^3_\mu A^\mu_+(A^3_zA^-_{\bar{z}} + A^3_{\bar{z}}A^-_z) + h.c.\right] - \left[A^\mu_+A^+_\mu A^-_{\bar{z}}A^-_z + \text{h.c.}\right]) \\
&+ ig[(\partial_\mu A^3_z - D_zA^3_\mu)(A^\mu_-A^+_{\bar{z}} - A^\mu_+A^-_{\bar{z}}) + (\partial_\mu A^+_z - D_zA^+_\mu)(A^\mu_3A^-_{\bar{z}} - A_{\bar{z}}{}^3A^\mu_-) \\
&+ (\partial_\mu A^-_z - D_zA^-_\mu)(A^\mu_+A_{\bar{z}}{}^3 - A^\mu_3A^+_{\bar{z}}) - \text{h.c.}] \\
&+ \partial_\mu A^+_\mu\,(D_zA^-_{\bar{z}} + D_{\bar{z}}A^-_z) + \partial_\mu A^-_\mu\,(D_zA^+_{\bar{z}} + D_{\bar{z}}A^+_z) + \partial_\mu A^3_\mu\,(D_zA^3_{\bar{z}} + D_{\bar{z}}A^3_z)\,.
\end{aligned} \tag{3.58}$$

From the $4D$ point of view, $\mathcal{L}_{\mu\nu}$, $\mathcal{L}_{ij}$ and $\mathcal{L}_{\mu i}$ will generate - after fixing the gauge - the pure gauge Lagrangian, the scalar potential and the gauge invariant kinetic terms of the scalar sector, respectively. Notice the term $2\mathcal{H}\,A^-_{\bar{z}}A^+_z$ in $\mathcal{L}_{ij}$: it corresponds to a negative mass squared for the A^+_z field, which pinpoints the instability of the theory expanded around a false vacuum.

Gauge fixing Lagrangian: the R^{6D}_ξ gauge

The structure of the $\mathcal{L}_{\mu i}$ term suggests immediately a certain gauge choice compatible with the boundary conditions, that we will call the R^{6D}_ξ gauge. Among all terms in the $6D$ Lagrangian containing two fluctuation fields, i.e. $\mathcal{L}^{(2)}_A$, the only $4D$ derivative interaction of the A_μ is of the form

$$-A^\mu_a\partial_\mu\left(D_zA^a_{\bar{z}} + D_{\bar{z}}A^a_z\right) , \tag{3.59}$$

and it appears in the last row of $\mathcal{L}_{\mu i}$. These terms are cancelled by the following choice for the gauge-fixing Lagrangian

$$\mathcal{L}^{g.f.}_{6\xi} = -\frac{1}{2\,\xi}\sum_a\,\left[\partial_\mu A^\mu_a\;-\;\xi\;(D_zA^a_{\bar{z}}\;+\;D_{\bar{z}}A^a_z)\right]^2 . \tag{3.60}$$

A warning is pertinent here. Not all terms which lead to $4D$ mixed terms (bilinears involving $4D$ derivatives of gauge fields and scalar fields) will be eliminated through this gauge choice. Additional $4D$ mixed terms may result from the cubic couplings appearing in the third and fourth rows of $\mathcal{L}_{\mu i}$, if some $4D$ scalars take vacuum expectation values due to the instability of the present expansion of the Lagrangian. In other words, the naive R_ξ^{6D} gauge defined above does not match a proper $4D$ R_ξ gauge. We will come back to this point later on, in subsection 3.4.

3.2.3 The effective 4-dimensional $SU(2)$ Lagrangian

The $4D$ Lagrangian,

$$\mathcal{L}^{4D} = \int_{\mathcal{T}_2} d^2y\, \mathcal{L}(x,y)\,, \tag{3.61}$$

will describe the physics of $4D$ fields, $A_M^{a\,(r)}(x)$, defined from

$$A_M^a(x,y) \equiv \sum_r A_M^{a\,(r)}(x) f^{a(r)}(y)\,, \tag{3.62}$$

with the extra-dimensional wave functions $f^{a(r)}$ satisfying the boundary conditions

$$\left\{ \begin{array}{lcl} f^{a(r)}(y_1 + l_1, y_2) & = & e^{i\pi(k+\frac{m}{N})\frac{y_2}{l_2}\,q^a}\; f^{a(r)}(y_1,y_2)\,, \\ f^{a(r)}(y_1, y_2 + l_2) & = & e^{-i\pi(k+\frac{m}{N})\frac{y_1}{l_1}\,q^a}\; f^{a(r)}(y_1,y_2)\,, \end{array} \right. \tag{3.63}$$

and r referring to the infinite towers of $4D$ modes. Depending on their gauge charge, fields are neutral ($a = 3$) or charged ($a = \pm$) with respect to the *imposed* background, and may be arranged in $4D$ KK towers ($r = n_1, n_2$) for the former and Landau levels ($r = j$) for the latter.

The shape of the extra-dimensional wave functions depends exclusively on the boundary conditions, encoded in the covariant derivative. That is, the wave functions depend on the gauge index (whether neutral or charged with respect to the background), but do not depend on its Lorentz index (whether $4D$ vectors or scalars).

Neutral fields

For neutral fields, the covariant derivatives D_i reduce to ordinary (commuting) derivatives. For the $4D$ vectors $A_\mu^{3\,(n_1,n_2)}(x)$, the following masses result

$$(\partial_z\partial_{\overline{z}} + \partial_{\overline{z}}\partial_z) f^{3\,(n_1,n_2)}(y) \;=\; m^2_{3\,(n_1,n_2)} f^{3\,(n_1,n_2)}(y)\,, \tag{3.64}$$

where

$$m^2_{3\,(n_1,n_2)} \equiv 4\pi^2 \left(\frac{n_1^2}{l_1^2} + \frac{n_2^2}{l_2^2}\right) , \tag{3.65}$$

while the eigenfunctions are given by

$$f^{3(n_1,n_2)}(y) = \frac{1}{\sqrt{\mathcal{A}}}\, e^{2\pi i\left(n_1 \frac{y_1}{l_1} + n_2 \frac{y_2}{l_2}\right)} . \tag{3.66}$$

The mode $A_\mu^{3\,(0,0)}(x)$ remains massless at this level, as it would for a residual $U(1)$ symmetry.

For neutral scalar fields, the quadratic mass terms in the R_ξ^{6D} gauge, Eqs. (3.57) and (3.60), lead to the following $4D$ Lagrangian after integration over the extra dimensions,

$$\left(\mathcal{L}^{4D}_{ij}\right)_2^{neutral} = -\frac{1}{2} \sum_{n_1,n_2=-\infty}^{\infty} m^2_{3\,(n_1,n_2)} \left\{ A^{(-n_1,-n_2)}(x)\, A^{(n_1,n_2)}(x) \,+\, \xi\, a^{(-n_1,-n_2)}(x)\, a^{(n_1,n_2)}(x) \right\} ,$$

where $A^{(n_1,n_2)}(x)$ and $a^{(n_1,n_2)}(x)$ are the mass eigenstates,

$$a^{(n_1,n_2)}(x) \;\equiv\; \frac{-i}{\sqrt{2}} \left(e^{i\theta_{(n_1,n_2)}} A_z^{3\,(n_1,n_2)}(x) + e^{-i\theta_{(n_1,n_2)}} A_{\bar z}^{3\,(-n_1,-n_2)}(x)\right) , \tag{3.67}$$

$$A^{(n_1,n_2)}(x) \;\equiv\; \frac{1}{\sqrt{2}} \left(e^{-i\theta_{(n_1,n_2)}} A_{\bar z}^{3\,(-n_1,-n_2)}(x) - e^{i\theta_{(n_1,n_2)}} A_z^{3\,(n_1,n_2)}(x)\right) , \tag{3.68}$$

with $e^{i\theta_{(n_1,n_2)}} \equiv \frac{2\pi}{m_{3(n_1,n_2)}} \left(\frac{n_1}{l_1} + i\frac{n_2}{l_2}\right)$.

In the absence of instability, $A^{(n_1,n_2)}(x)$ would be the physical neutral scalar fields, while $a^{(n_1,n_2)}(x)$ would play the role of pseudo-Goldstone bosons, eaten by the $A_\mu^{3\,(n_1,n_2)}(x)$ to acquire mass. Notice that indeed the quantity $D_z A^3_{\bar z} + D_{\bar z} A^3_z$ appearing in the gauge fixing condition, Eq. (3.60), can be expressed in terms of the scalars $a^{(n_1,n_2)}$ alone:

$$D_z A^3_{\bar z} + D_{\bar z} A^3_z = -\sum_{n_1,n_2=-\infty}^{\infty} m_{3(n_1,n_2)}\, a^{(n_1,n_2)}(x)\, f^{(n_1,n_2)}(y) . \tag{3.69}$$

Notice as well that it does not exist a pseudo-Goldstone boson with $n_1 = n_2 = 0$, which is consistent with the fact that $A_\mu^{3\,(0,0)}$ has not received, at this level, a contribution to its mass.

Charged fields

To determine the Landau energy levels, define as usual creation and destruction operators a and $a^\dagger$, for charges $q^\pm = \pm 2$,

$$\begin{array}{llll} a_+ \equiv -\frac{i}{\sqrt{2\mathcal{H}}}\, D_{\bar z}^{(+)} , & & a_- \equiv \frac{i}{\sqrt{2\mathcal{H}}}\, D_z^{(-)} , \\ a_+^\dagger \equiv -\frac{i}{\sqrt{2\mathcal{H}}}\, D_z^{(+)} , & & a_-^\dagger \equiv \frac{i}{\sqrt{2\mathcal{H}}}\, D_{\bar z}^{(-)} , \end{array} \tag{3.70}$$

which satisfy commutation relations

$$\left[a_\pm, a_\pm^\dagger\right] = 1\ . \tag{3.71}$$

Defining as well the number operator $\hat{j}_{(\pm)} = a^\dagger_{(\pm)} a_{(\pm)}$, it results that charged fields $A_M^{\pm (j)}(x)$ get at least partial contributions to their masses from the term

$$-(D_z^a D_{\bar{z}}^a + D_{\bar{z}}^a D_z^a)\, f^{a(j)}(y) = m^2_{a\,(j)}\, f^{a(j)}(y)\,, \tag{3.72}$$

with $a = \pm$ and mass eigenvalues given by

$$m^2_{\pm\,(j)} \equiv 2\mathcal{H}(2j+1) \ = \ \frac{4\pi(k+\frac{m}{2})}{\mathcal{A}}\,(2j+1)\,, \tag{3.73}$$

where j integer ≥ 0.

That is, for charged fields the commutator in Eq. (3.44) does not vanish and in consequence no zero eigenvalues are expected. In other words, while neutral fields can be simultaneously at rest with respect to the two extra dimensions, charged fields cannot, as a charged particle in a magnetic field moves. The energy levels are Landau levels. Notice as well that the mass scale is set by the torus area, the 't Hooft non-abelian flux m and the integer k, while it is independent of the $6D$ coupling constant g.

The associated extra-dimensional wave functions,

$$\begin{aligned} f^{+(j,\rho)}(x,y) \ &= \ \left(\frac{2\,d}{l_1^3 l_2}\right)^{\frac{1}{4}} \frac{(-i)^j}{\sqrt{2^j\, j!}}\, e^{i\pi\, d\, \frac{y_1 y_2}{l_1\, l_2}} \quad \times \\ &\sum_{n=-\infty}^{\infty} e^{-\frac{\pi d}{l_1 l_2}(y_2+nl_2+\frac{\rho l_2}{d})^2}\, e^{2\pi i (d\,n+\rho)\frac{y_1}{l_1}}\, H_{j,\rho}\left[\sqrt{\frac{2\pi d}{l_1\, l_2}}\left(y_2+nl_2+\frac{\rho l_2}{d}\right)\right] \end{aligned} \tag{3.74}$$

are derived explicitly in Appendix A. The opposite-charge field is $f^{-(j,\rho)}(x,y) \ = \ \left(f^{+(j,\rho)}(x,y)\right)^*$. Obviously, $f^{+(j,\rho)}$ and $f^{-(j,\rho)}$ satisfy the boundary conditions in Eq. (3.63).

The quantity d in Eq. (3.74) is defined by

$$d \equiv q\,(k+\frac{m}{N})\,, \tag{3.75}$$

and signals degeneracy. Notice the index ρ: generically, the tower of Landau levels may be defined by another quantum number [106] in addition to j. ρ sweeps over these extra degrees of freedom,

$$0 \ \leq \ \rho \ \leq \ d-1\,, \tag{3.76}$$

and its possible values signal degenerate energy levels, as the latter are independent of ρ, see Eq. (3.73) above. For a field of given charge q (i.e, $q = 2$ and $q = 1$ for fields in the adjoint and fundamental representation of $SU(2)$, respectively), the degree of degeneracy is given by d. As discussed in Appendix A, d is necessarily an integer, which for $SU(2)$ reduces to either $d = qk$ or $d = q(k + \frac{1}{2})$, depending on the value of m.

While $4D$ charged vectors $A_\mu^{\pm(j,\rho)}$ get only mass contributions from Eq. (3.73) above, charged scalars receive further contributions from quadratic terms in Eq. (3.57). Working in the R_ξ^{6D} gauge, Eq. (3.60), and, after diagonalizing the system, we obtain

$$\begin{aligned}\left(\mathcal{L}_{ij}^{4D}\right)_2^{charged} \quad = \quad & \sum_{\rho=0}^{d-1} \left\{ 2\mathcal{H}\, H_{0,\rho}^*(x) H_{0,\rho}(x) - 2\mathcal{H} \sum_{j=1}^{\infty} (2j+1) H_{j,\rho}^*(x)\, H_{j,\rho}(x) \right. \\ & \left. -\xi\, 2\mathcal{H} \sum_{j=0}^{\infty} (2j+1) h_{j,\rho}^*(x)\, h_{j,\rho}(x) \right\} . \end{aligned} \tag{3.77}$$

This Lagrangian has been written in terms of the following mass eigenfunctions:

$$\begin{aligned} & H_{0,\rho}(x) = -A_{\overline{z}}^{-\,(0,\rho)}(x)\,, \\ & h_{0,\rho}(x) = A_{\overline{z}}^{-\,(1,\rho)}(x)\,, \\ & H_{j,\rho}(x) = -s_j A_{\overline{z}}^{-\,(j+1,\rho)}(x) + c_j A_z^{-\,(j-1,\rho)}(x)\,, \\ & h_{j,\rho}(x) = c_j A_{\overline{z}}^{-\,(j+1,\rho)}(x) + s_j A_z^{-\,(j-1,\rho)}(x)\,, \end{aligned} \tag{3.78}$$

where $c_j \equiv \cos\theta_j = \sqrt{\frac{j+1}{2j+1}}$ and $s_j \equiv \sin\theta_j = \sqrt{\frac{j}{2j+1}}$, with $j \geq 1$. $H_{0,\rho}(x)$ denotes the $4D$ field (or fields, when ρ takes several values) with negative mass(es) $-2\mathcal{H}$ and $h_{0,\rho}(x)$ its unphysical scalar partner(s), eaten -at this level- by the $A_\mu^{+\,(0,\rho)}(x)$ field(s) to become massive[7].

In the absence of the instability induced by the negative mass, $H_{j,\rho}(x)$ would be the physical charged scalar fields, while $h_{j,\rho}(x)$ would play the role of pseudo-Goldstone bosons, eaten by the $A_\mu^{+\,(j,\rho)}(x)$ fields to acquire mass. Indeed, the gauge fixing condition can be expanded as

$$D_z A_{\overline{z}}^- + D_{\bar{z}} A_z^- = i \sum_{\rho=0}^{d-1} \sum_{j=1}^{\infty} m_{\pm j}\, h_{j,\rho}(x)\, f^{-(j,\rho)}(y)\,. \tag{3.79}$$

Notice as well that this result holds for any value of j, including $j = 0$, since ${A_\mu}^{\pm\,(0,\rho)}(x)$ has taken a contribution to its mass after compactification, as a consequence of its interaction with the imposed background.

[7]The tachyon $H_{0,\rho}$ could also be correctly denoted $H_{-1,\rho}$, as a $j = -1$ state, extending the definition given for the $H_{j,\rho}$ fields. We have refrained from doing so, though, with the aim of beautifying the notation.

The Lagrangian exhibits thus a behavior that could correspond to the breaking $SU(2) \rightarrow U(1)$, although the presence of the tachyon $H_{0,\rho}(x)$ signals that the true vacuum remains to be found. The remaining analysis can be technically simplified working in the R_ξ^{6D} gauge with $\xi = \infty$: the would-be Goldstone fields $a(x)$ and $h(x)$ disappear then from the analysis, and results will be given for this case. However, before proceeding to it, let us briefly discuss another gauge-fixing choice, alternative to that used above.

3.2.4 The R_ξ^{4D} gauge

An appropriate gauge choice, also compatible with the boundary conditions, is

$$\mathcal{L}_{4\xi}^{g.f.} = -\frac{1}{2\,\xi} \sum_a \left[\partial_\mu A_a^\mu - \xi \left(\widetilde{D}_z A_{\bar{z}}^a + \widetilde{D}_{\bar{z}} A_z^a\right)\right]^2 , \tag{3.80}$$

where now $\widetilde{D}_i$ is the *total* covariant derivative defined in Eq. (3.52), corresponding to a stable background. Notice the analogy with the analysis in the previous subsections in terms of the R_ξ^{6D} gauge, Eq. (3.60). The choice in Eq. (3.80) guarantees the elimination of **all** $4D$ scalar-gauge crossed terms stemming from the last three rows of $\mathcal{L}_{\mu i}$, Eq. (3.58), including those resulting after spontaneous symmetry breaking. It is then a true R_ξ gauge from the four-dimensional point of view.

In this gauge, it is trivial to formally identify the terms in the 6D Lagrangian which will give rise to the masses of the different type of $4D$ fields: gauge bosons and their replica, physical scalars and "would be" Goldstone bosons:

1. Gauge boson masses will result from

$$\mathcal{L}_{mass}^{gauge} = -\frac{1}{2}\, A_\mu^a \left[\widetilde{D}_z\, \widetilde{D}_{\bar{z}} + \widetilde{D}_{\bar{z}}\, \widetilde{D}_z\right]_{ab} A^{\mu\, b} , \tag{3.81}$$

where a, b are the indices in the adjoint representation.

2. Physical, ξ-independent, scalar masses will stem from

$$\begin{aligned} \mathcal{L}_{mass}^{scal} &= -\frac{1}{2} \left(\widetilde{D}_z A_{\bar{z}}^a - \widetilde{D}_{\bar{z}} A_z^a\right)^2 \\ &= -\frac{1}{2}\, (A_z^a,\, A_{\bar{z}}^a) \begin{pmatrix} -\widetilde{D}_{\bar{z}}\widetilde{D}_{\bar{z}} & \widetilde{D}_{\bar{z}}\widetilde{D}_z \\ \widetilde{D}_z\widetilde{D}_{\bar{z}} & -\widetilde{D}_z\widetilde{D}_z \end{pmatrix}_{ab} \begin{pmatrix} A_z^b \\ A_{\bar{z}}^b \end{pmatrix} . \end{aligned} \tag{3.82}$$

Because $[\widetilde{D}_z, \widetilde{D}_{\bar{z}}] = 0$ (see Eq. (3.51)), the eigenvalues of this matricial equation produce the following mass contributions to scalar fields:

$$\begin{aligned} \Delta M_{physical}^2 &= \tfrac{1}{2} \left[\widetilde{D}_z\widetilde{D}_{\bar{z}} + \widetilde{D}_{\bar{z}}\widetilde{D}_z\right] , \\ \Delta M_{goldstone}^2 &= 0 . \end{aligned} \tag{3.83}$$

Comparison with Eq. (3.81) shows that it is generically expected to find a scalar partner for each 4D gauge boson, degenerate in mass.

3. Finally, the ξ-dependent scalar masses will result from,

$$\begin{aligned}
\mathcal{L}^{\xi}_{mass} &= -\frac{\xi}{2}\left(\widetilde{D}_z A^a_{\bar{z}} + \widetilde{D}_{\bar{z}} A^a_z\right)^2 \\
&= \frac{1}{2}\left(A^a_z,\, A^a_{\bar{z}}\right)\begin{pmatrix} \widetilde{D}_{\bar{z}}\widetilde{D}_{\bar{z}} & \widetilde{D}_{\bar{z}}\widetilde{D}_z \\ \widetilde{D}_z\widetilde{D}_{\bar{z}} & \widetilde{D}_z\widetilde{D}_z \end{pmatrix}_{ab}\begin{pmatrix} A^b_z \\ A^b_{\bar{z}} \end{pmatrix}.
\end{aligned} \tag{3.84}$$

Once again, because $\widetilde{D}_z$ and $\widetilde{D}_{\bar{z}}$ commute, the eigenvalues of $\mathcal{L}^{\xi}_{mass}$ will result in mass contributions

$$\begin{aligned}
\Delta M^2_{goldstone} &= \frac{\xi}{2}\left[\widetilde{D}_z\widetilde{D}_{\bar{z}} + \widetilde{D}_{\bar{z}}\widetilde{D}_z\right], \\
\Delta M^2_{physical} &= 0\,.
\end{aligned} \tag{3.85}$$

The coincidence between the eigenvalues expected for the gauge and "would be" goldstone boson masses is a characteristic of hidden non-abelian symmetries. The larger degeneracy among the three sectors -gauge, physical scalars and unphysical scalars- is related to the fact that *total* field strength of the stable vacuum is zero. In consequence the coordinate dependent conditions are equivalent to constant ones, as shown in Section 2, which discriminate among gauge charges, not among Lorentz indices.

In the next Section, we will follow the dynamical evolution of the system towards a stable vacuum, determining the minimum of the $4D$ potential and obtaining the physical spectra in both the R^{4D}_{ξ} and R^{6D}_{ξ} gauges.

3.3 The minimum of the 4-dimensional potential

Below, we will obtain the effective $4D$ potential for $SU(2)$, minimize it and find the physical spectra. The results will be compared with the theoretical expectations developed in Section 2.

We have first integrated the $6D$ Lagrangian, Eqs. (3.56)-(3.58), plus the gauge-fixing term, Eq. (3.60) or Eq. (3.80), over the $2D$ torus surface, obtaining in this way all effective $4D$ couplings among the towers of states. In ordinary compactifications, i.e. without background with constant field strength, a good understanding of the $4D$ light spectrum only requires to consider the lightest KK states and their self-interactions. With the inclusion of such background, this is no more the case due to the simultaneous presence of KK and Landau levels. Cubic and quartic terms link a given neutral (KK) field to an infinity of charged (Landau) levels, and viceversa. Previous analysis of scenarios with background with constant field strength, such as the original Nielsen and Olesen one [71–73], as well as subsequent studies [78], have typically included only quartic interactions of the lowest $4D$ charged level (i.e. the tachyon), with at most the addition of the tower of only one type of replica. However, we will show that it is necessary to consider many modes and all types of interaction between KK and Landau levels, for a true understanding of the system.

For quadratic terms, the integration over the torus reduces to the use of the orthogonality relations for the bases of extra-dimensional wave functions. The inclusion of cubic and quartic interactions requires the evaluation of integrals with three and four extra-dimensional wave functions. We have solved them analytically in the general case. The results can be found in Appendix B, together with the completeness relationships linking them. The latter have been checked as well numerically up to a precision better than 10^{-6}.

We have then proceeded to look for the minima of the potential. Let us previously recall the theoretical expectations. As the true vacuum should have total zero energy, see Eq. (3.17), the stable minimum of the $SU(2)$ $4D$ potential should correspond to a dynamical reaction of the system of the form

$$F^3_{12}(x,y)\big|_{min} = -G^3_{12} \;=\; \frac{2\mathcal{H}}{g} \;=\; \frac{4\pi}{g\,\mathcal{A}}(k+\frac{m}{2})\,, \tag{3.86}$$

so as to cancel the contribution of the *imposed* background. That is, the following value for the minimum of the $4D$ potential is expected (see Eq.(3.16)):

$$V\big|_{min} = \frac{1}{2}\int_{\mathcal{T}^2} dy\,[(F^3_{12}(x,y))^2 + 2\,G^3_{12}\,F^3_{12}(x,y)]\;\big|_{min} \;=\; -\frac{8\pi^2}{g^2\mathcal{A}}\,(k+\frac{m}{2})^2\,. \tag{3.87}$$

We analyze below whether the minimum of our $4D$ effective potential does converge towards such values. Three comments on the procedure are pertinent:

1. The determination of the set of vacuum expectation values that minimizes the potential can only be done numerically. Starting with the inclusion of only the lightest fields of the KK and Landau towers, heavier replicas of both types will be successively added and the corresponding minimum determined at each step. The total number of neutral and charged replica to be included in the analysis is determined requiring that the minimum of the potential reaches an asymptotically stable regime.

2. For technical and theoretical reasons, we will present our results in the two gauges previously described: the R^{6D}_ξ gauge, for the particular case $\xi=\infty$, and the general R^{4D}_ξ gauge. This will allow precise checks of the gauge invariance of the results.

3. In order to keep as low as possible the degeneracy of states, while analyzing the two possible non-trivial setups, the numerical results will be confined to two cases: a)

$m = 0\,,\ k = 1$ and b) $m = 1\,,\ k = 0$. Furthermore, all numerical results presented below correspond to an isotropic torus[8], $l_1 = l_2$.

3.3.1 Non-trivial 't Hooft flux: $m = 1$, $k = 0$

This case corresponds to a non-trivial 't Hooft flux, in which the generators of the translation operators T_i anti-commute. The fields in the Landau towers are not degenerate, as $d = 1$ in Eq. (3.75): the index ρ become thus meaningless and it will be obviated all through this Subsection.

Let us illustrate with a simple argument how the system dynamically approaches the true vacuum and the need of including rather high neutral and charged modes. Consider for the moment only the charged scalar zero mode, H_0 (i.e. the tachyon), the lightest neutral scalar $A_z^{3\,(0,0)}$ and their interactions. The effective $4D$ potential is then simply given by:

$$V \;=\; -2\mathcal{H}\,|H_0(x)|^2 + \frac{g^2}{2}\,I_0^{(4)}\,|H_0(x)|^4 + |H_0(x)|^2\,A_z^{3\,(0,0)}(x)\,A_{\bar{z}}^{3\,(0,0)}(x)\,, \qquad (3.88)$$

with $I_0^{(4)}$ referring to the 4-point integral between the lightest charged states[9]. One can immediately recognize in Eq. (3.88) the classical mexican-hat potential, with its minimum corresponding to:

$$<|H_0(x)|^2> \;=\; \frac{2\mathcal{H}}{g^2 I_0^{(4)}}\,, \qquad <A_z^{3\,(0,0)}(x)>=<A_{\bar{z}}^{3\,(0,0)}(x)> \;=\; 0\,. \qquad (3.89)$$

In this simplified example, only the charged scalar (i.e.the tachyon) acquires a non vanishing vacuum expectation value (*vev*) while the neutral fields remain unshifted. Using the numerical value $1/I_0^{(4)} = (0.85\,\mathcal{A})$, it results[10]:

$$V_{min} = -\frac{2\mathcal{H}^2}{g^2 I_0^{(4)}} \sim -0.85 \times \frac{2\pi^2}{g^2 \mathcal{A}}\,, \qquad (3.90)$$

which is still quite different from that predicted by Eq. (3.87). Moreover, it is enough to add the interactions with either the next neutral or charged levels to observe the appearance of tadpole terms. That is, the true minimum of the system does not correspond then anymore to the *vev*s obtained in Eq. (3.89), but all fields get new shifts instead.

[8]The anisotropic case will be considered in a future work.

[9]The general definition of the 3-point and 4-point integrals is given in Appendix B. Here $I_0^{(4)}$ is an abbreviated notation for the integral $I_C^{(4)}[0,0,0,0,0,0,0,0]$ defined there.

[10]The dimensions of the quantities in Eq. (3.89) are $[\mathcal{H}] = [I_0^{(4)}] = [E^2]$ and $[g] = [E^{-1}]$.

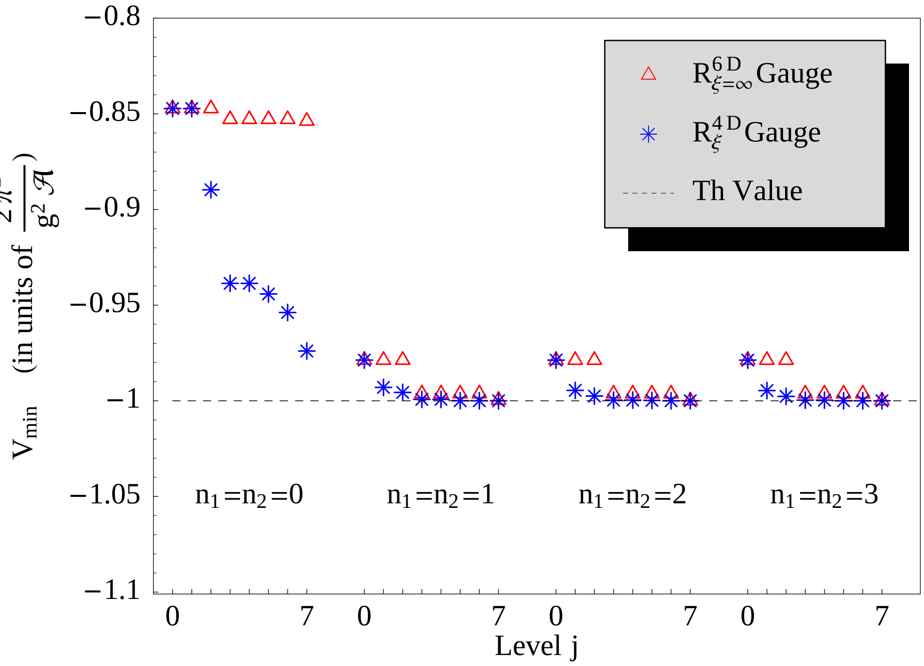

Figure 3.1: *Values of the minimum of the scalar potential as heavier degrees of freedom are included. Triangles (stars) represent the numerical results obtained in the $R^{6D}_{\xi=\infty}$ (R^{4D}_{ξ}) gauge. The horizontal dashed line represents the theoretically predicted value for the potential minimum, in the non-trivial 't Hooft flux case.*

We found that generically all charged and neutral fields in the two towers get *vevs*. Fig. (3.1) shows the dynamical approach to the true minimum by the successive addition of heavier charged modes (labelled by $j = 0, \cdots, 7$ in the horizontal axis) and heavier neutral modes (labelled with $n_1 = n_2 = 0, \cdots, 3$), for both the R^{4D}_{ξ} and $R^{6D}_{\xi=\infty}$ gauges. For example, the point labelled with $n_1 = n_2 = 1$ and $j = 3$ represents the numerical calculation where **all** degrees of freedom up to $n_1 = n_2 = 1$ and $j = 3$ are included. The graphic shows that the value of the minimum of the scalar potential does converge to the theoretically predicted value of $-2\pi^2/(g^2\mathcal{A})$: for $n_1 = n_2 \geq 1$ (≥ 5 neutral complex fields) and $j \geq 3$ (≥ 4 charged complex fields) a precision over 1% is achieved, in both gauges; for $n_1 = n_2 = 3$ and $j = 7$, it reaches 10^{-5} (10^{-7}) for the $R^{6D}_{\xi=\infty}$ (R^{4D}_{ξ}) gauge.

As regards the symmetries of the spectrum, the numerical results confirm that the $SU(2)$ symmetry is completely broken. This is well illustrated by Fig. 3.2, where the lightest vector state is shown to be asymptotically massive. The horizontal dashed line represents the mass value of 0.25 (in units of $4\pi^2/\mathcal{A}$), theoretically predicted in Eq.(3.35). An excellent agreement is observed as well between the calculations in the two gauges after the levels up to $n_1 = n_2 \geq 1$ and $j \geq 3$ are included. We have thus explicitly proved

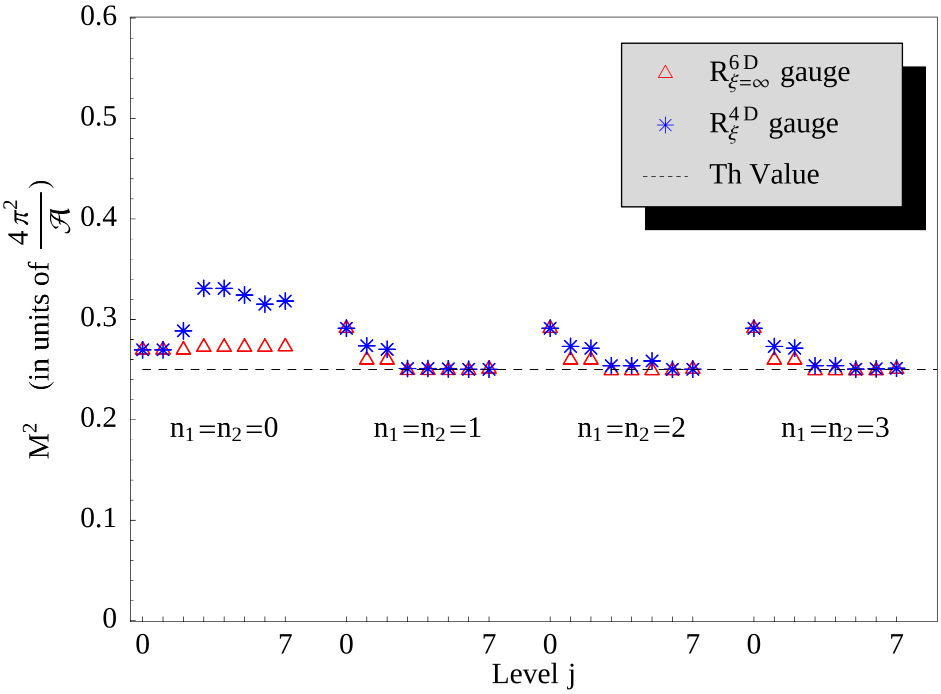

Figure 3.2: *Lightest gauge mode mass. Triangles (stars) represent the numerical results obtained in the $R^{6D}_{\xi=\infty}$ (R^{4D}_{ξ}) gauge. The horizontal dashed line represents the theoretically predicted value in the non-trivial 't Hooft flux case.*

that the $SU(2)$ symmetry is completely broken.

In Fig. 3.3 the full spectrum of the $4D$ vector fields is displayed, with all fields up to $n_1 = n_2 = 3$ and $j = 7$ included in the estimation, in the R^{4D}_{ξ} and $R^{6D}_{\xi=\infty}$ gauges. No visible difference can be noticed. This result is a strong numerical proof of the consistency of our effective $4D$ Lagrangian, and its manifest gauge invariance when a sufficient number of heavy degrees of freedom are included.

Finally, Fig. 3.4 retakes the full spectrum, resulting from the diagonalization of the complete system, in the R^{4D}_{ξ} gauge: gauge bosons (stars), physical scalars (empty triangles) and unphysical scalars (full triangles), with the latter corresponding to the choice $\xi = 0$. Superimposed, the Figure shows as well (black dots joined by a full line) the theoretical prediction for constant discrete Scherk-Schwarz boundary conditions, Eq.(3.35). Notice that:

- Each $4D$ vector boson has a physical scalar partner degenerate in mass, as expected in the asymptotic limit from Eqs.(3.81) and (3.83).

- The unphysical scalar spectrum -which constitutes half of the scalar spectrum- is identified as those fields which appear to have zero mass, as expected for "pseudo-

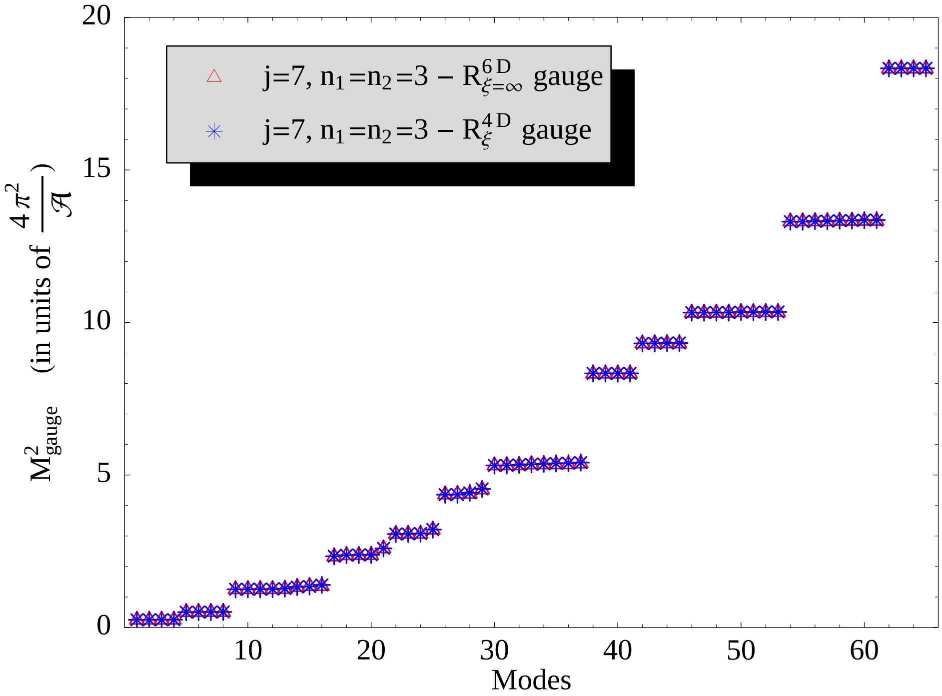

Figure 3.3: *Gauge invariance of the gauge spectrum for the non-trivial 't Hooft flux case. Triangles (stars) represent the numerical results obtained in the $R^{6D}_{\xi=\infty}$ (R^{4D}_{ξ}) gauge respectively, for $n_1 = n_2 = 3$ and $j = 7$.*

goldstone bosons" eaten by the vector fields to acquire masses[11]. A slight numerical mismatch only appears for the masses of the pseudo-goldstone fields of the heavier modes, as the numerical truncation of the tower of states starts to be felt.

- The coincidence between the numerical results -obtained with y-dependent boundary conditions- and the spectrum predicted for constant discrete Scherk-Schwarz boundary conditions (black dots) is very good up to the first 20 modes (i.e. around $M^2 \approx 3$ in the units chosen for illustration). The agreement of the overall scale, as well as the expected four-fold degeneracy of the first two massive levels and the eight-fold degeneracy of the next one, are clearly seen. Only the higher levels start to show disagreement with the theoretical formulas. This is as it should be, as the present numerical analysis was restricted to charged levels up to $j = 7$ and neutral ones up to $n_1 = n_2 = 3$. Indeed, the next mode non-included in the numerical

[11]As stated, this numerical spectrum has been computed for $\xi = 0$, but it can also be viewed as corresponding to the ξ-independent contributions to the goldstone masses for any ξ, as it follows from Eq. (3.83).

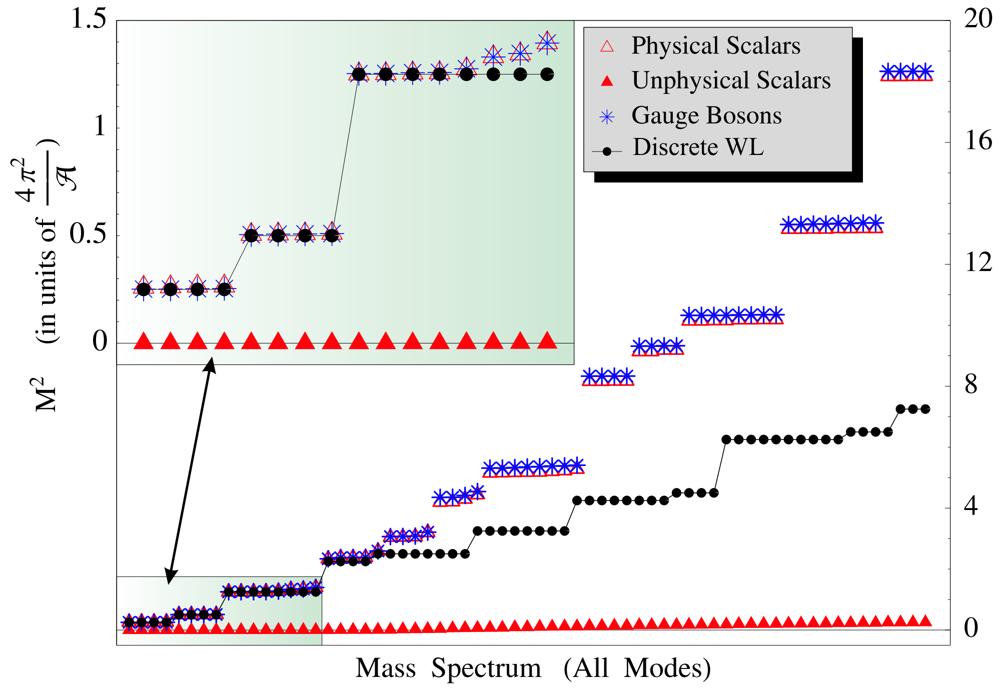

Figure 3.4: *Full spectrum for the non-trivial 't Hooft flux case, in the $R^{4D}_{\xi=0}$-gauge. Gauge bosons (stars), physical scalars (empty triangles) and unphysical scalars (full triangles) are shown. The minimization procedure includes all charged and neutral modes up to $n_1 = n_2 = 3$ and $j = 7$. Black dots joined by a full line represent the theoretically predicted masses derived in Section 2.2.*

analysis would be $j = 8$, which has a squared mass $M^2 \approx 2.7$. In consequence, the numerical results and the theoretical prediction start to diverge around this scale. The mode $j = 8$ sets the limit of validity of the present numerical analysis, while a better agreement can be reached including higher modes.

We have also computed the physical spectrum in the R^{4D}_{ξ} gauge by another procedure: the direct substitution of the *vevs* obtained from the numerical minimization into the *total* covariant derivatives in Eqs. (3.81) and (3.83). The coincidence with the numerical results shown above is so precise that it would be indistinguishable within the drawing precision.

3.3.2 Trivial 't Hooft flux: $m = 0$, $k = 1$

Consider now the case of trivial 't Hooft flux, in which the generators of the translation operators T_i commute. The simplest non-trivial configuration of this type[12] corresponds

[12]That is, with lowest degeneracy.

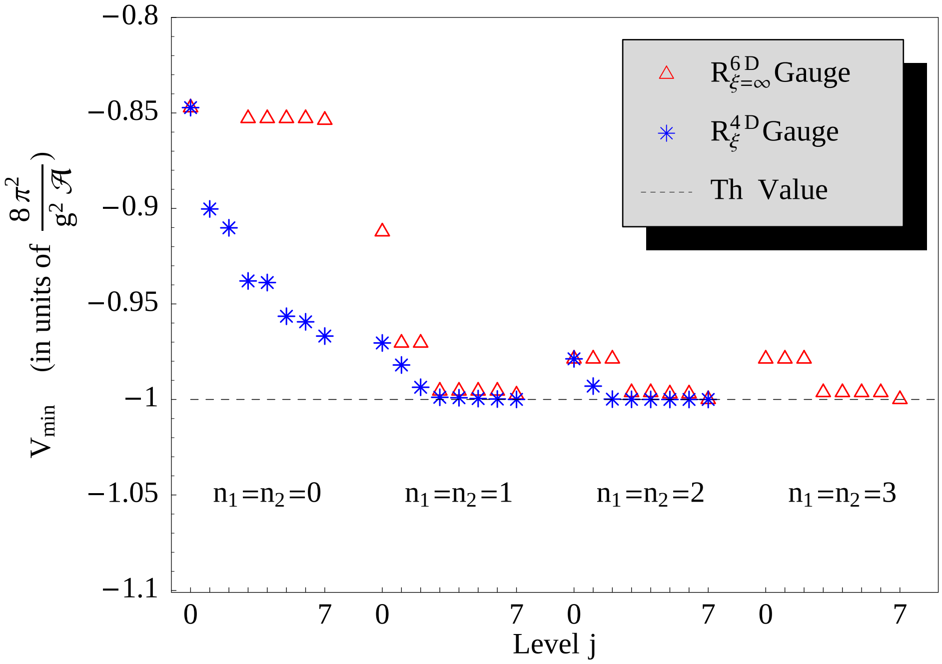

Figure 3.5: *Values of the minimum of the scalar potential as heavier degrees of freedom are included. Triangles (stars) represent the numerical results obtained in the $R^{6D}_{\xi=\infty}$ (R^{4D}_{ξ}) gauge. The horizontal dashed line represents the theoretically predicted value for the potential minimum, in the trivial 't Hooft flux case.*

to $m = 0$ and $k = 1$. A two-fold degeneracy of the charged (Landau) levels is then present, as $d = 2$ in Eq. (3.75) and $\rho = 0, 1$. In consequence, due to the higher number of states, the numerical treatment is more cumbersome than in the previous Subsection.

The dynamical approach to the minimum of the $4D$ potential can be seen in Fig. 3.5. Again it shows how the asymptotic regime is reached with the successive addition of heavier charged and neutral fields. The dashed horizontal line represents the theoretical predicted value, $-8\pi^2/g^2\mathcal{A}$, as expected from Eq.(3.87): for $n_1 = n_2 \geq 1$ (≥ 5 neutral fields) and $j \geq 3$ (≥ 4 charged fields) a precision over 1% is achieved, both in the $R^{6D}_{\xi=\infty}$ gauge and in the R^{4D}_{ξ} gauge. In the best case that we could numerically evaluate for the $R^{6D}_{\xi=\infty}$ gauge ($n_1 = n_2 = 3$, $j = 7$), a precision of $\mathcal{O}(10^{-5})$ has been obtained.

As regards the expected spectra, recall from Subsection 2.1 that all possible solutions should correspond to either unbroken $SU(2)$ symmetry or a $SU(2) \to U(1)$ breaking patterns, all of them being degenerate in the absence of quantum corrections and fermions. All numerical results obtained here turn out to correspond to $SU(2) \to U(1)$ breaking examples. This is well illustrated by Fig. 3.6 where the mass of one (and only one) vector

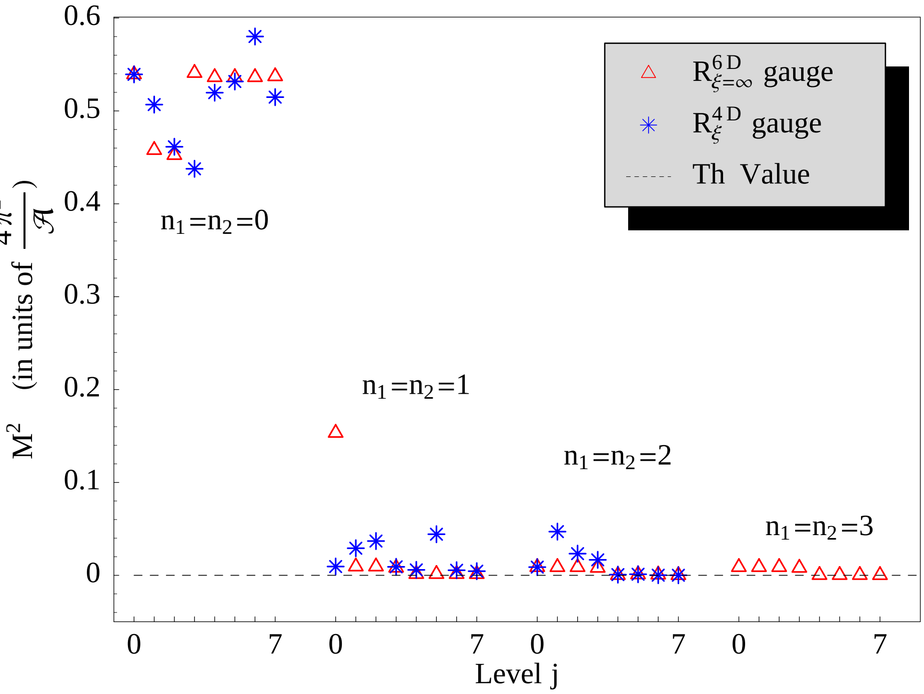

Figure 3.6: *Lightest gauge mode mass. Triangles (stars) represent the numerical results obtained in the $R^{6D}_{\xi=\infty}$ (R^{4D}_{ξ}) gauge. The horizontal dashed line represents the theoretically predicted value in the trivial 't Hooft flux case.*

state is seen to vanish asymptotically, in agreement with the lightest value predicted in Eq.(3.25) for $\alpha_i \neq 0$. That state is the $4D$ gauge vector boson of the unbroken $U(1)$ symmetry. The figure also shows clearly that if only the first few light levels of the KK and Landau towers would have been considered in the analysis, the lightest state would have looked massive, suggesting a fake $SU(2) \to \varnothing$ breaking pattern. Only the inclusion of higher charged and neutral levels allows to attain the asymptotic regime, unveiling then the remaining $U(1)$ symmetry. Numerically, the agreement with the theoretical prediction starts to be satisfactory for $n_1 = n_2 \geq 1$ and $j \geq 3$, analogously to the case with non-trivial 't Hooft flux in the previous Subsection.

It is worth pointing out that the $U(1)$ symmetry of the *total* stable vacuum selects, in general, a different gauge direction, in $SU(2)$ space, than that of the *imposed* abelian background. In other words, it may be a different $U(1)$ symmetry than that naively exhibited by the Lagrangian, when expanded around the *imposed* background. The neutral and charged towers of fields, as defined by the latter, have recombined dynamically, to select the final stable symmetric direction.

Fig. 3.7 shows two gauge spectra obtained numerically including all modes up to $n_1 = n_2 = 2$ and $j = 7$, for the two gauges $R^{6D}_{\xi=\infty}$ (triangles) and R^{4D}_{ξ} (stars). Notice

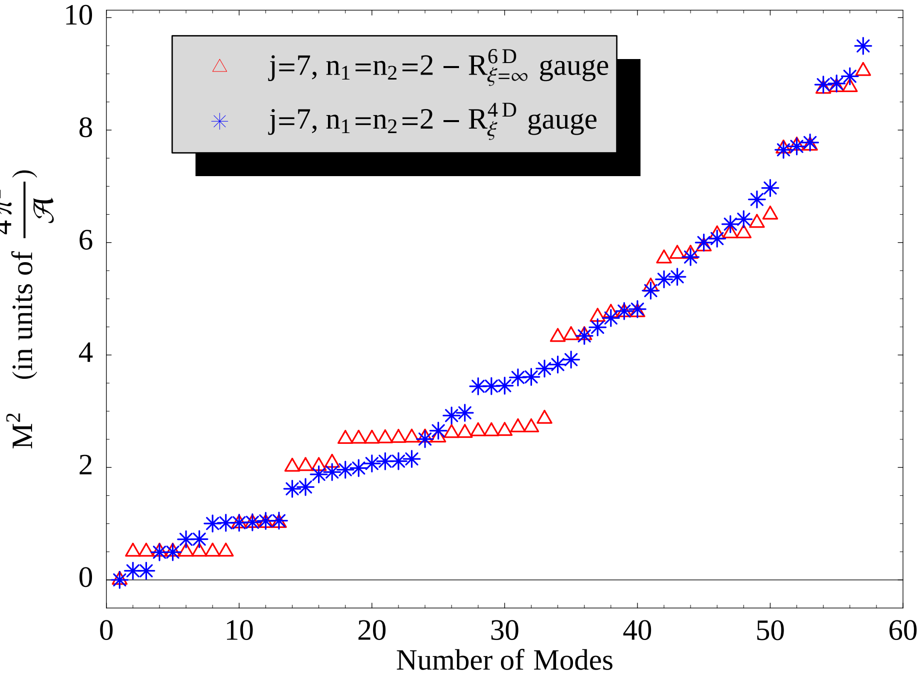

Figure 3.7: *Gauge boson spectra for the trivial 't Hooft flux case. Triangles (stars) represent the numerical results obtained in the* $R^{6D}_{\xi=\infty}$ *(*R^{4D}_{ξ}*) gauge respectively, for* $n_1 = n_2 = 2$ *and* $j = 7$*. In this example, the two spectra turn out to correspond to different sets of* (α_1, α_2) *values:* $(1/2, 1/2)$ *(triangles) and* $(0.33, 0.22)$ *(stars).*

the difference with the analogous figure obtained for the $m = 1$ case, Fig. 3.3: at first sight, one could think that the test of gauge invariance fails in the present case. This is not the case, though: the two spectra turn out to correspond to different values for the set of arbitrary parameters α_1, α_2, in Eq. (3.27), which parametrize the possible Scherk-Schwarz spectra. We determined the values chosen by the minimization algorithm in these examples, performing a two-parameter fit to the first 20 masses obtained from the numerical procedure. The χ^2 value of the fit is extremely significant for both gauges. It resulted in the values $\alpha_1 = \alpha_2 = 1/2$ for the example shown in the $R^{6D}_{\xi=\infty}$ gauge, as can be easily deduced from the observed boson multiplicity. Conversely, for the R^{4D}_{ξ} gauge calculation, the minimization algorithm selected $\alpha_1 = 0.334$ and $\alpha_2 = 0.219$, to which it corresponds the observed lower multiplicity of degenerate fields. Examples corresponding to other values have also been obtained, although not illustrated here. The existence of different spectra for the same symmetry breaking pattern is generic of Scherk-Schwarz compactification at the classical level.

In Fig. 3.8 we retake the gauge (stars), physical scalar (empty triangles) and unphysical

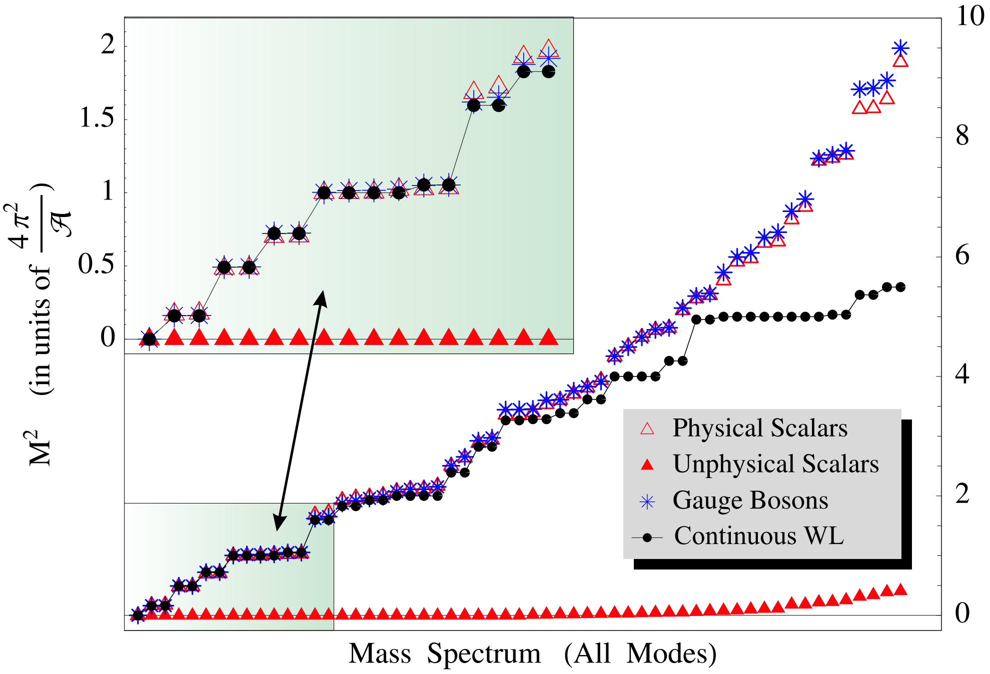

Figure 3.8: *Numerical results for the trivial 't Hooft flux case, in the* R_ξ^{4D}*-gauge. Gauge bosons (stars), physical scalars (empty triangles) and unphysical scalars (stars) are shown. The minimization procedure includes all the charged and neutral modes up to* $n_1 = n_2 = 2$ *and* $j = 7$*. Black dots joined by a full line represent the theoretically predicted masses derived in Section 2.1, for the case* $\alpha_1 = 0.33$, $\alpha_2 = 0.22$.

scalar (full triangles) spectra, in the R_ξ^{4D} gauge, for the same α_i values than in the previous figure, and with the unphysical scalar masses computed for $\xi = 0$. Due to the degeneracy of the Landau levels, the numerical analysis could only be performed including modes up to $n_1 = n_2 = 2$ and $j = 7$. The masses of the unphysical scalar degrees of freedom tend, as before, to vanish -as they should- as the asymptotic regime is approached. For the heavier modes, a slight numerical mismatch appears between the masses of the vector fields and those of their physical scalar partners. A corresponding tiny mass for the unphysical scalar partners is also observed. This discrepancy is again consequence of the truncation error. Apart form this subtlety, physical scalar and gauge masses are in excellent agreement.

Moreover, the agreement between the numerical spectra and the theoretically predicted one - typical of Scherk-Schwarz breaking and represented in Fig. 3.8 with black dots joined by a full line - is very good up to the first 40 modes (i.e.around $M^2 \approx 4$ in the units chosen). This scale sets the validity limit for the present numerical analysis of our low-energy effective 4D theory. A better agreement above this scale could be obtained adding higher modes. Once again, the mass of the next non-included mode, the $j = 8$ mode, is $M^2 \approx 5.4$ and coincides with the scale at which the numerical masses and the theoretical predicted ones start to diverge.

Finally, we have also computed the physical spectrum in the R_ξ^{4D} gauge by another procedure: the direct substitution of the *vev*s obtained from the numerical minimization into the *total* covariant derivatives in Eqs. (3.81) and (3.83). The coincidence with the numerical results shown above is so precise that it would be indistinguishable within the drawing precision.

In summary, in this Section we have thus explicitly shown, for the $6D$ $SU(2)$ gauge group compactified on a $2D$ torus, that a stable vacuum of zero energy is reached, out of the initial unstable configuration. To solve the system with y-dependent boundary conditions has been shown to be tantamount to solve it with constant boundary conditions. For the case of non-trivial 't Hooft flux, the pattern of symmetry breaking obtained is $SU(2) \longrightarrow \varnothing$ and it corresponds to Scherk-Schwarz symmetry breaking with discrete Wilson lines. For trivial 't Hooft flux, the patterns found correspond to $SU(2) \longrightarrow U(1)$ and are equivalent to Scherk-Schwarz symmetry breaking with continuous Wilson lines.

3.4 Conclusions and outlook

Boundary conditions depending upon the extra coordinates are equivalent to constant ones, for $SU(N)$ on a two-dimensional torus. For trivial 't Hooft flux, they are equivalent to constant Scherk-Schwarz boundary conditions, associated to continuous Wilson lines. For the case of non-trivial 't Hooft flux, the coordinate-dependent boundary conditions can be traded instead by constant Scherk-Schwarz boundary conditions, associated to discrete Wilson lines, resulting always in symmetry breaking. One of the novel features of this work is the study of the phenomenological implications of this last scenario, studying the pattern of gauge symmetry breaking and the spectrum of the four-dimensional vector and scalar excitations.

Chirality cannot be implemented within a $SU(N)$ background and will require to consider in the future non-simply connected groups. For them, the equivalence between coordinate-dependent and constant boundary conditions does not hold in general. A field-theory treatment of the system subject to coordinate dependent boundary conditions is then necessary to solve the details of the four-dimensional spectrum. We start this approach in the present work by treating also explicitly the case of $SU(2)$ on a torus with background.

We have explicitly solved the Nielsen-Olesen instability on the two dimensional torus.

For the obtention of the four-dimensional effective Lagrangian, all couplings have been taken into account, including *all* quartic and cubic terms mixing Kaluza-Klein and Landau levels. Those terms are shown to be essential in the determination of the stable minimum of the potential and its symmetries. The corresponding integrals over the extra-dimensional space have been obtained analytically for all modes, for the first time. Furthermore, we have defined gauge-fixing Lagrangians, appropriate when both Kaluza-Klein and Landau levels are simultaneously present and interacting. We found that the naive R_ξ gauge defined in six dimensions is then *not* equivalent to the R_ξ gauge in four dimensions. The computations have been performed in different possible gauge choices and the issue has been clarified in depth. These technical tools will be necessary when groups other than $SU(N)$ will be considered.

The system is seen to evolve dynamically from the unstable background configuration towards a stable and non-trivial background of zero energy. This happens through an infinite chain of vacuum expectation values of the four-dimensional scalar fields. The resulting spectra do show explicitly the symmetries expected from the theoretical analysis mentioned above, for the case of $SU(N)$ with constant boundary conditions.

It turns out that for each four-dimensional gauge boson there exists a scalar partner degenerate in mass, both for trivial and non-trivial 't Hooft fluxes. This is one of the important phenomenological drawbacks that the approach has to face. The scenario has to be enlarged then, for instance including more than just one scale in the theory. Indeed, a motivation for the present work was the hypothetical identification of the Higgs field as a component of a gauge boson in full space, which would make its mass insensitive to ultraviolet contributions, unlike in the Standard Model. To find a realistic pattern of electroweak symmetry breaking, which matches the spectra found in nature, remains a non-trivial issue.

Chapter 4

Symmetry breaking from generalized Scherk-Schwarz compactification

We analyze the classical stable configurations of an extra-dimensional gauge theory, in which the extra dimensions are compactified on a torus. Depending on the particular choice of gauge group and the number of extra dimensions, the classical vacua compatible with four-dimensional Poincaré invariance and zero instanton number may have zero energy. For $SU(N)$ on a two-dimensional torus, we find and catalogue all possible degenerate zero-energy stable configurations in terms of continuous or discrete parameters, for the case of trivial or non-trivial 't Hooft non-abelian flux, respectively. We then describe the residual symmetries of each vacua.

The main purpose of this chapter is to find and classify all possible vacua and to describe the residual symmetries, for the general case of a $SU(N)$ gauge theory on a two-dimensional torus, for both the cases of trivial and non-trivial 't Hooft non-abelian flux. More in detail, in section 4.1 we provide a novel method to analyze the vacuum energy of a general Lie group on an even-dimensional torus. For the case of $SU(N)$ on T^2, we re-obtain a well-known result [81]: the stable vacua have always zero energy, including the case with coordinate-dependent periodicity conditions. In section 4.2, we discuss the relation between coordinate-dependent and constant transition functions and we find under which conditions they are equivalent. For $SU(N)$, such result will allow to introduce in section 4.3 the background symmetric gauge. In this gauge, we find and classify all the stable vacua and describe their symmetries for the case of trivial 't Hooft non-abelian flux as well as for the non-trivial case. Finally, in section 4.4, we conclude. The Appendix C includes supplementary arguments and develops technical tools.

4.1 Vacua of $SU(N)$ on T^2

Consider a $SU(N)$ gauge theory on a $\mathcal{M}_4 \times T^2$ space-time. In what follows, we will denote by x the coordinates of the four-dimensional Minkowski space $\mathcal{M}_4$ and by y the extra space-like dimensions.

A gauge field living on T^2 has to be periodic up to a gauge transformation under the fundamental shifts $\mathcal{T}_a : y \to y + l_a$ with $a = 1, 2$, that define the torus[1]:

$$\begin{aligned} \mathbf{A_M}(x, y + l_a) &= \Omega_a(y)\mathbf{A}_M(x, y)\Omega_a^\dagger(y) + \frac{i}{g}\Omega_a(y)\partial_M\Omega_a^\dagger(y) && (4.1) \\ \mathbf{F}_{MN}(x, y + l_a) &= \Omega_a(y)\mathbf{F}_{MN}(x, y)\Omega^\dagger(y)\,, && (4.2) \end{aligned}$$

where $M, N = 0, 1, ..., 5$, $a = 1, 2$ and l_a is the length of the direction a. The eqs.(4.1)-(4.2) are known as coordinate dependent Scherk-Schwarz compactification. The transition functions $\Omega_a(y)$ are the embedding of the fundamental shifts in the gauge space and in order to preserve four-dimensional Poincaré invariance, they can only depend on the extra dimensions y. Under a gauge transformation $S \in SU(N)$, the $\Omega_a(y)$ transform as

$$\Omega_a'(y) = S(y + l_a)\,\Omega_a(y)\,S^\dagger(y)\,. \tag{4.3}$$

The transition functions are constrained by the following consistency condition coming from the geometry:

$$\Omega_1(y + l_2)\,\Omega_2(y) \;=\; e^{2\pi i \frac{m}{N}}\,\Omega_2(y + l_1)\,\Omega_1(y)\,. \tag{4.4}$$

The factor $\exp[2\pi i m/N]$ is the embedding of the identity in the gauge space[2]. The gauge invariant quantity $m = 0, 1, .., N - 1$ is a topological quantity called *non-abelian 't Hooft flux* [68].

The total Hamiltonian for a $SU(N)$ theory on a $\mathcal{M}_4 \times T^2$ space-time, reads

$$\begin{aligned} H &= \frac{1}{2}\int_{\mathcal{M}_4} d^4x \int_{T^2} d^2y \;\; \mathrm{Tr}\left[\mathbf{F}^{MN}\mathbf{F}_{MN}\right] \\ &= \frac{1}{2}\int_{\mathcal{M}_4} d^4x \int_{T^2} d^2y\, \mathrm{Tr}\left[\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu} + \mathbf{F}^{\mu a}\mathbf{F}_{\mu a} + \mathbf{F}^{ab}\mathbf{F}_{ab}\right]\,, && (4.5) \end{aligned}$$

where here and in what follows, $\mu, \nu = 0, 1, 2, 3$ and a, b denote the extra coordinates. Since we are interested in configurations with $\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu} = 0$ and which preserve four-dimensional

[1] For simplicity, we consider an orthogonal torus, but all the results can be generalized to a non orthogonal T^2.

[2] A non-trivial value of m is possible in the absence of field representations sensitive to the center of the group.

Poincaré invariance (that is $\mathbf{F}_{\mu a}(x,y) = 0$ and $\mathbf{F}_{ab}(x,y) = \mathbf{F}_{ab}(y)$), to minimize the expression in eq. (4.5) reduces to minimize the quantity

$$H_{T^2} = \frac{1}{2}\int_{T^2} d^2y \ \operatorname{Tr}\left[\mathbf{F}^{ab}(y)\,\mathbf{F}_{ab}(y)\right] \geq 0\,. \tag{4.6}$$

The latter inequality follows from the fact that we are working on an Euclidean manifold.

We will show that the vacuum energy is always zero, *i.e.* $\langle \mathbf{F}_{ab} \rangle = 0$, including the case of coordinate-dependent periodicity conditions. This result reflects the non-existence of topological quantities for a $SU(N)$ gauge theory on a T^2.

Let us consider the issue for the more general case of a Lie gauge group $\mathcal{G}$ on an even dimensional torus (T^{2n} with the integer $n \geq 1$), in order to pinpoint the dependence of the result on the choice of the gauge group and of the number of extra dimensions.

Parametrize the $(4+2n)$-dimensional gauge field $\mathbf{A}_M$ as

$$\left\{\begin{array}{lcl} \mathbf{A}_\mu(x,y) &=& A_\mu(x,y) \\ \mathbf{F}_{\mu\nu}(x,y) &=& F_{\mu\nu}(x,y) \end{array}\right. , \quad \left\{\begin{array}{lcl} \mathbf{A}_a(x,y) &=& B_a(y) + A_a(x,y) \\ \mathbf{F}_{ab}(x,y) &=& G_{ab}(y) + F_{ab}(x,y) \end{array}\right. , \tag{4.7}$$

where the background $B_a(y)$ has the following properties:

i) It is a solution of the $2n$ dimensional Yang-Mills equations of motion.

ii) It has non-trivial field strength.

iii) It is compatible with the periodicity conditions of eqs.(4.1)-(4.2).

$A_\mu(x,y)$, $A_a(x,y)$ are the fluctuations fields. The background and fluctuation field strengths are defined as

$$\begin{array}{rcl} G_{ab} &=& \partial_a B_b - \partial_b B_a - ig\left[B_a, B_b\right]\,, \\ F_{\mu\nu} &=& \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]\,, \\ F_{ab} &=& D_a A_b - D_b A_a - ig\left[A_a, A_b\right]\,. \end{array} \tag{4.8}$$

In eq. (4.8), $D_a A_b$ denotes the background covariant derivative

$$D_a A_b = \partial_a A_b - ig\left[B_a, A_b\right]\,, \tag{4.9}$$

satisfying

$$\left[D_a,\, D_b\right] \;=\; -i\,g\,G_{ab}\,. \tag{4.10}$$

Now $a, b = 1, ..., 2n$. For what follows, notice that for a non-simple gauge group, a solution of the classical Yang-Mills equations of motion can be associated to generators belonging to the normal subgroup of the algebra. Such background B_a satisfies $[B_a, A_b] = 0$ and, therefore, the covariant derivatives with respect to it reduce to ordinary derivatives.

Generalizing the discussion in ref. [74,107], we diagonalize the background field strength with respect to the Lorentz indices. The first step is to perform an appropriate $O(2n)$ rotation able to write the $2n \times 2n$ matrix $G_{ab}(y)$ as[3]

$$G_{ab} = \left(\begin{array}{cccc|cccc} & & & & f_1(y) & 0 & ... & 0 \\ & & 0 & & 0 & f_2(y) & ... & 0 \\ & & & & ... & ... & ... & ... \\ & & & & 0 & 0 & .. & f_n(y) \\ -f_1(y) & 0 & ... & 0 & & & & \\ 0 & -f_2(y) & ... & 0 & & 0 & & \\ ... & ... & ... & ... & & & & \\ 0 & 0 & .. & -f_n(y) & & & & \end{array} \right) , \tag{4.11}$$

where $f_i(y)$ for $i = 1, ..., n$ are matrices belonging to the adjoint representation of the gauge group $\mathcal{G}$. The second step is to introduce the complex basis $\{z_i, \overline{z}_i\}$ defined as

$$z_i = \frac{1}{\sqrt{2}} (y_i + \, i\, y_{n+i}) \, , \qquad \overline{z}_i = \frac{1}{\sqrt{2}} (y_i - \, i\, y_{n+i}) \, , \tag{4.12}$$

for $i = 1, ..., n$. In this basis, the background field strength is diagonal in the Lorentz space

$$G_{ab} = \mathrm{Diag}\left[i f_1(z), -i f_1(z), i f_2(z), -i f_2(z), ..., i f_n(z), -i f_n(z) \right] , \tag{4.13}$$

and the commutators between the covariant derivatives in eq. (4.10), reduce to

$$\begin{aligned} \left[D_{z_i}, D_{z_j} \right] &= \left[D_{\overline{z}_i}, D_{\overline{z}_j} \right] = 0 \, , \\ \left[D_{z_i}, D_{\overline{z}_j} \right] &= g\, f_i(z)\, \delta_{ij} \quad , \qquad \forall\, i, j = 1, ..., n \, . \end{aligned} \tag{4.14}$$

We introduce the following gauge fixing Hamiltonian compatible with the $2n$-dimensional generalization of the periodicity conditions in eqs.(4.1)-(4.2)

$$H_{g.f.} = \int_{T^{2n}} d^n z \, d^n \overline{z} \ \mathrm{Tr} \left[\sum_{i=1}^{n} D_{z_i} A^{\overline{z}_i} + D_{\overline{z}_i} A^{z_i} \right]^2 \, . \tag{4.15}$$

Denote $H_{T^{2n}}$ the $2n$-dimensional generalization of the Hamiltonian in eq. (4.6). Using eq. (4.7), the expansion of $H_{T^{2n}} + H_{g.f.}$, up to second order in the perturbation fields $A_a(x, y)$, reads

$$H_{T^{2n}} + H_{g.f.} = H^{(1A)} + H^{(2A)} + \mathcal{O}\left(A^3\right)$$

[3]It follows from the fact that on an Euclidean flat space as T^{2n}, the non-trivial coordinate dependence of G_{ab} is completely determined only by the gauge indices as it can be proved using the periodicity conditions of eqs.(4.1)-(4.2) and the Yang-Mills equations of motion on a flat space.

where

$$H^{(1A)} \;=\; -2\sum_{i=1}^{n}\int_{T^{2n}} d^n z\, d^n\overline{z}\ \mathrm{Tr}\left[A^{z_i}D^{\overline{z}_i}G_{z_i\overline{z}_i} \,+\, A^{\overline{z}_i}D^{z_i}G_{\overline{z}_i z_i}\right] \ , \tag{4.16}$$

$$H^{(2A)} \;=\; \sum_{i=1}^{n}\int_{T^{2n}} d^n z\, d^n\overline{z}\ \mathrm{Tr}\left[A^{z_i}\mathcal{M}^2_{\overline{z}_i z_i}A^{\overline{z}_i} + A^{\overline{z}_i}\mathcal{M}^2_{z_i\overline{z}_i}A^{z_i}\right] \ . \tag{4.17}$$

The operators $\mathcal{M}^2_{z_i\overline{z}_i}$ and $\mathcal{M}^2_{\overline{z}_i z_i}$ in eq. (4.17) are given by

$$\mathcal{M}^2_{\overline{z}_i z_i} \;\equiv\; \sum_{k=1}^{n}\Sigma_k \,+\, 2\,\Gamma_i \ , \tag{4.18}$$

$$\mathcal{M}^2_{z_i\overline{z}_i} \;\equiv\; \sum_{k=1}^{n}\Sigma_k \,-\, 2\,\Gamma_i \,, \tag{4.19}$$

where

$$\Sigma_i \;\equiv\; -\left\{D_{z_i},\, D_{\overline{z}_i}\right\} \ , \tag{4.20}$$

$$\Gamma_i \;\equiv\; \left[D_{z_i},\, D_{\overline{z}_i}\right] \ , \qquad\qquad \forall i = 1,...,n\,. \tag{4.21}$$

The background B_a is then seen to be stable if and only if it is stationary, *i.e.* $H^{(1A)} = 0$, and the eigenvalues of the operators defined in eqs.(4.18)-(4.19) are all semi-positive.

Since B_a is a solution of the classical equations of motion, it is stationary by construction.

In order to discuss the sign of the eigenvalues of the operators in eqs.(4.18)-(4.19), we recall that

- $\forall i = 1,..,n$, the operators Σ_i are defined semi-positive:

$$\Sigma_i = -D_{z_i}D_{\overline{z}_i} - D_{\overline{z}_i}D_{z_i} \;=\; |D_{z_i}|^2 + |D_{\overline{z}_i}|^2 \geq 0\,, \tag{4.22}$$

  since $(D_{z_i})^\dagger = -D_{\overline{z}_i}$ and $(D_{\overline{z}_i})^\dagger = -D_{z_i}$.

- The background B_a satisfies the Yang-Mills equations of motion and then the operators Σ_i, Γ_i commute. Consequently, there exists a basis that diagonalizes simultaneously (with respect to the gauge indices) these operators. We denote with $|\,\lambda_{\Sigma_i}, \lambda_{\Gamma_i}\,\rangle$ the elements of such basis satisfying

$$\begin{aligned}\Sigma_k \ \ |\lambda_{\Sigma_i}, \lambda_{\Gamma_i}\,\rangle &\;=\; \lambda_{\Sigma_k}\ |\lambda_{\Sigma_i}, \lambda_{\Gamma_i}\,\rangle \ ,\\ \Gamma_k \ \ |\lambda_{\Sigma_i}, \lambda_{\Gamma_i}\,\rangle &\;=\; \lambda_{\Gamma_k}\ |\lambda_{\Sigma_i}, \lambda_{\Gamma_i}\,\rangle \ ,\end{aligned}$$

  for any $k = 1,..,n$.

We start analyzing the eigenvalues of the operators of eqs.(4.18)-(4.19) associated to the elements $|\lambda_{\Sigma_i}, \lambda_{\Gamma_i}\rangle$ belonging to the subspace characterized by $\lambda_{\Gamma_i} = 0\ \forall i = 1, .., n$, that is to the subspace in which $[D_{z_i}, D_{\overline{z}_i}] = 0$, $\forall i = 1, .., n$. All the elements of this subspace have semi-positive defined eigenvalues since eqs.(4.18)-(4.19) reduce to

$$\mathcal{M}^2_{\overline{z}_i z_i} = \mathcal{M}^2_{z_i \overline{z}_i} \quad \equiv \quad \sum_{k=1}^{n} \Sigma_k \geq 0\,. \tag{4.23}$$

Notice that for the case of a non simple gauge group with background such that $[B_a, A_b] = 0$, the subspace $\lambda_{\Gamma_i} = 0$ coincides with the whole space.

Consider, now, the subspace associated to eigenvalues $\lambda_{\Gamma_i} \neq 0$. It can be analyzed using the analogy with the harmonic oscillator, *i.e.* using the non-trivial commutation rules in eq. (4.21). The vacuum $|0\rangle$ is characterized by

$$\begin{array}{rlll} -D_{\overline{z}_i} D_{z_i}|0\rangle & = 0 & \text{if} & \lambda_{\Gamma_i} < 0 \\ -D_{z_i} D_{\overline{z}_i}|0\rangle & = 0 & \text{if} & \lambda_{\Gamma_i} > 0 \\ -D_{z_i} D_{\overline{z}_i}|0\rangle = -D_{\overline{z}_i} D_{z_i}|0\rangle & = 0 & \text{if} & \lambda_{\Gamma_i} = 0\,. \end{array} \tag{4.24}$$

For simplicity, we will discuss explicitly the subspace associated to the elements for which all $\lambda_{\Gamma_i} \neq 0$ are positive[4]. The vacuum is, therefore, defined

$$-D_{z_i} D_{\overline{z}_i}|0\rangle = 0\,, \tag{4.25}$$

for all i associated to $\lambda_{\Gamma_i} \geq 0$. Introduce the notation $\Sigma_i|0\rangle = \lambda^0_{\Sigma_i}|0\rangle$ and $\Gamma_i|0\rangle = \lambda^0_{\Gamma_i}|0\rangle$. Since $-D_{z_i} D_{\overline{z}_i} = 1/2\,(\Sigma_i - \Gamma_i)$, eq. (4.25) implies

$$\lambda^0_{\Sigma_i} = \lambda^0_{\Gamma_i}\,. \tag{4.26}$$

The eigenvalues of the operators in eqs.(4.18)-(4.19) associated to the vacuum $|0\rangle$ read

$$\mathcal{M}^2_{\overline{z}_i z_i}\,|0\rangle \;=\; \left(\sum_{k=1}^{n} \lambda^0_{\Sigma_k} + 2\lambda^0_{\Gamma_i}\right)|0\rangle\,, \tag{4.27}$$

$$\mathcal{M}^2_{z_i \overline{z}_i}\,|0\rangle \;=\; \left(\sum_{k=1}^{n} \lambda^0_{\Sigma_k} - 2\lambda^0_{\Gamma_i}\right)|0\rangle\,. \tag{4.28}$$

Since $\lambda_{\Sigma_k} \geq 0$ for any $k = 1, ..., n$, the right hand side (RHS) of eq. (4.27) is always positive for $\lambda_{\Gamma_i} > 0$. On the contrary, the sign of the eigenvalue in the RHS of eq. (4.28) is not determined *a priori* for the general case of a Lie gauge group $\mathcal{G}$ on T^{2n}.

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[4]The subspaces associated to eigenstates for which some $\lambda_{\Gamma_i} < 0$ can be obtained from the following reasoning interchanging $z_i \leftrightarrow \overline{z}_i$ for those indices i such that $\lambda_{\Gamma_i} < 0$.

Focusing on the case of $SU(N)$ on T^2, that is $n = 1$, eq. (4.28) reduces to

$$\mathcal{M}^2_{z_i\bar{z}_i}\,|0\rangle \;=\; -\lambda^0_{\Gamma_i}\,|0\rangle \qquad \text{with} \quad \lambda^0_{\Gamma_i} > 0\,. \tag{4.29}$$

In this case, a background with a non-trivial field strength is, therefore, always unstable, since the operators defined in eqs.(4.18)-(4.19) always admit at least one negative eigenvalue. On the other side, all stable background configurations necessarily must have zero field strength, *i.e.* $\lambda_{\Gamma_i} = 0$.

Notice that such result depends on the choice of the gauge group ($SU(N)$) and of the number of dimensions of the torus (T^2).

Change for example the gauge group, considering instead $U(N)$ on a T^2. $U(N)$ is a non-simple group and, as we have discussed before, it is possible to consider solutions of the equations of motion with non-trivial field strength pointing to the internal direction associated to the identity. In this case the background covariant derivatives defined in eq. (4.9), reduce to the ordinary ones and consequently they commute. The operators $\mathcal{M}^2_{z_i\bar{z}_i}$, $\mathcal{M}^2_{\bar{z}_i z_i}$ are then given by the expressions in eq.(4.23) and, therefore, are semi-positive defined. In this case, it is, therefore, possible to have stable background with non-trivial field strength. Notice that these stable configurations have non-zero energy and are classified by some non-trivial topological charge: in this case the first Chern class.

Change now, instead, the number of dimensions of the torus. Consider for example $SU(N)$ on T^4 ($n = 2$). In this case, eq. (4.28) reduces to

$$\begin{aligned} \mathcal{M}^2_{z_1\bar{z}_1}\,|0\rangle &= \left(\lambda^0_{\Gamma_2} - \lambda^0_{\Gamma_1}\right)\,|0\rangle \\ \mathcal{M}^2_{z_2\bar{z}_2}\,|0\rangle &= \left(\lambda^0_{\Gamma_1} - \lambda^0_{\Gamma_2}\right)\,|0\rangle\,. \end{aligned}$$

Unlike for $SU(N)$ on T^2, it is possible to have non-negative eigenvalues if the relation

$$\lambda^0_{\Gamma_1} = \lambda^0_{\Gamma_2} \tag{4.30}$$

is fulfilled. Changing the number of torus dimensions, stable background configurations with non-trivial field strength can thus exist [70, 108]. Notice though, that although the background field strength is non-trivial, the energy can be zero. The stable configurations with non-zero energy are classified by some non zero topological charge: in this particular case, the second Chern class.

4.2 Coordinate dependent vs constant transition functions

In the previous section, we have provided a novel demonstration of the fact that, on a two-dimensional torus, only non-simple gauge groups admit stable configurations with non-zero energy. In particular, for the case of $SU(N)$ on T^2 we have shown that all stable configurations are *flat connections*, that is configurations characterized by $\mathbf{F}_{ab} = 0$ and thus zero energy. A flat connection is a pure gauge configuration[5] given by

$$B_a = \frac{i}{g} U(y) \partial_a U^\dagger(y) \,. \tag{4.31}$$

Substituting eq. (4.31) into eq. (4.1), it follows that the $SU(N)$ gauge transformation $U(y)$ has to satisfy the following periodicity conditions

$$U(y + l_a) = \Omega_a(y)\, U(y)\, V_a^\dagger \,, \tag{4.32}$$

where $\Omega_a(y)$ are the transition functions solution of eq. (4.4), while the V_a's are constant elements of $SU(N)$ constrained by the consistency conditions

$$V_1\, V_2 \;=\; e^{2\pi i \frac{m}{N}}\, V_2\, V_1 \,. \tag{4.33}$$

Two pairs V_1, V_2 and V_1', V_2' are called *non-equivalent* if they are not connected by a $SU(N)$ gauge transformation. Notice that, given the transition functions $\Omega_1(y), \Omega_2(y)$, for each *non-equivalent* pair of V_1, V_2 there exists a different gauge transformation $U(y)$ satisfying eq. (4.32) and, therefore, a different zero-energy background B_a.

In this section, we investigate the conditions (choice of the gauge group, number of space-like dimensions) which guarantee that eq. (4.32) admit *always* a solution regardless of the choice of Ω_a and V_a. For this purpose, it is sufficient to understand when the 't Hooft consistency conditions only allow *equivalent* classes of solutions. We leave for the next section the task of classifying and describing all non-equivalent pairs of V_1, V_2.

As in the previous section, the proof will be carried through for the general case of a Lie gauge group $\mathcal{G}$ and a T^{2n} manifold. In this case, the 't Hooft consistency conditions read

$$\Omega_a(y + l_b)\, \Omega_b(y) \quad = \quad \mathcal{Z}_{ab}\, \Omega_b(y + l_a)\, \Omega_a(y) \,, \tag{4.34}$$

where $\mathcal{Z}_{ab}$ is the embedding of the identity in the gauge space, that is:

$$\forall g \in \mathcal{G} \qquad \mathcal{Z}_{ab}\, g \;=\; g\, \mathcal{Z}_{ab} \;=\; g \,. \tag{4.35}$$

Since Ω_a have to commute up to a factor that plays the role of identity, it follows from eq. (4.34) that the transition functions Ω_a have to satisfy the following periodicity conditions:

$$\Omega_a(y + l_b) \quad = \quad g_a^{(b)}(y)\, \Omega_a(y) \,, \tag{4.36}$$

[5]Here we adopt the same approach and notation used for the theoretical discussion in ref. [109].

where the phases $g_a^{(b)}(y)$ are constrained to verify

$$g_b^{(a)\,-1}(y)\, g_a^{(b)}(y) \quad = \quad \mathcal{Z}_{ab}\,. \tag{4.37}$$

For a gauge group $\mathcal{G}$ on a $2n$-dimensional torus, all transition functions, solutions of 't Hooft consistency conditions, are equivalent if and only if the gauge group $\mathcal{G}$ is $(2n-1)$-connected, i.e. the first $(2n-1)$ homotopy groups of $\mathcal{G}$ are trivial: $\Pi_i\,(\mathcal{G}) = 0$.

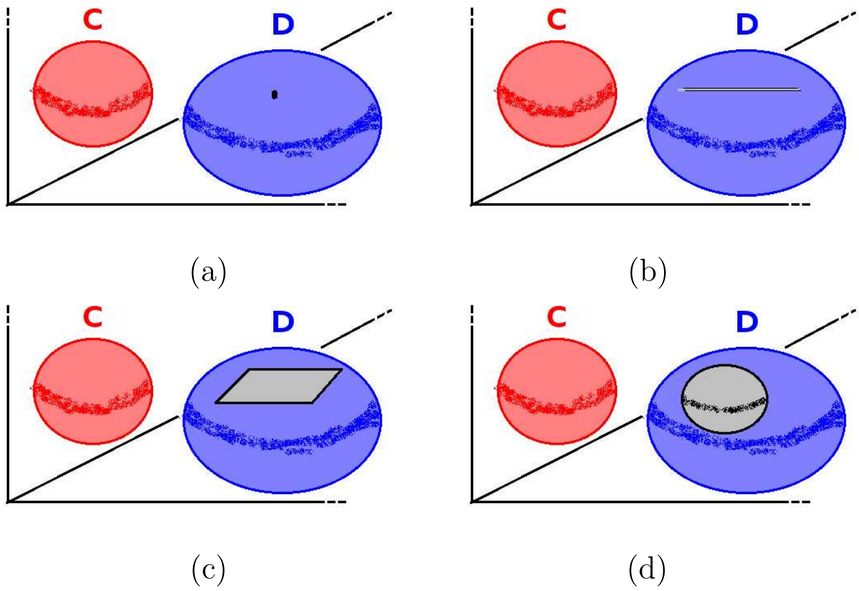

Figure 4.1: Examples of 3-dimensional topological spaces containing 0, 1, 2 and 3$-$dimensional defects (holes). In all cases, the presence of holes avoids to obtain the 3-dimensional ball **C** from the 3-dimensional ball **D** by continuous deformations. Notice that, although in the case (a) **D** can be continuously deformed to a point, the latter does not belong to the space.

Proof: Let $\Omega_a, \Omega_a^0 \in \mathcal{G}$, $a = 1, ..., 2n$, be generic (constant or not) sets of solutions of the consistency conditions in eq. (4.34). Treating $\mathcal{G}$ as a topological space, Ω_a (Ω_a^0) can be seen as $2n$ points of such topological space and thus describing a $2n$-dimensional ball $\mathcal{S}_{2n}$ ($\mathcal{S}_{2n}^0$).

To understand if two sets of solutions of the 't Hooft consistency conditions are equivalent, it is tantamount to determine when $\mathcal{S}_{2n}$ can be obtained from $\mathcal{S}_{2n}^0$ by a continuous deformation, *i.e.* when $\mathcal{S}_{2n}$ and $\mathcal{S}_{2n}^0$ are homotopic. In particular, all $2n$-dimensional balls contained in a topological space are homotopic *if and only if* they can always be shrunk to a point, see fig. 4.1. This result implies that the gauge group $\mathcal{G}$ as a topological space must not contain any j-dimensional defects (holes) with $j = 1, ..., 2n$ or, in other words, that $\mathcal{G}$ has to be $(2n-1)$-connected: $\Pi_i(\mathcal{G}) = 0$, $\forall i = 1, ..., 2n-1$.

The previous reasoning can be re-formulated in a more precise way as follows. We consider the product of transformations changing step by step a given Ω_a into a Ω_a^0:

$$U(y) = \prod_{r=1}^{2n} U_r(y) \,. \tag{4.38}$$

By construction, therefore, $U_1(y) \in \mathcal{G}$ transforms $\Omega_1 \to \Omega_1^0$, $U_2(y) \in \mathcal{G}$ transforms $U_1(y+l_2)\,\Omega_2 U_1^\dagger(y) \to \Omega_2^0$ and leaves invariant Ω_1^0, $U_3(y) \in \mathcal{G}$ transforms $U_2(y+l_3)\,U_1(y+l_3)\,\Omega_3 U_1^\dagger(y)\,U_2^\dagger(y) \to \Omega_3^0$ and leaves invariant Ω_1^0, Ω_2^0, etc...

Suppose that all U_r, with $r < \overline{r}$ and fixed $\overline{r} \in [1, 2n]$, exist regardless of the choice of Ω_a and Ω_a^0. We want to show that the existence of $U_{\overline{r}}(y)$ is necessary and sufficient for the absence of $\overline{r}$-dimensional holes in the gauge group $\mathcal{G}$, seen as a topological space.

The transformation $U_{\overline{r}}$ is defined as the transformation that allows

$$\Omega_{\overline{r}} \quad \overset{U_{\overline{r}}(y)}{\Longrightarrow} \quad \Omega_{\overline{r}}^0 \,, \tag{4.39}$$

and that leaves invariant all Ω_r^0 with $r < \overline{r}$. Such a gauge transformation has to satisfy the following periodicity conditions

$$\begin{aligned} U_{\overline{r}}(y+l_r) &= \Omega_r^0\, U_{\overline{r}}(y)\, \Omega_r^{0\,-1} \,, \qquad \forall \ r < \overline{r} && (4.40)\\ U_{\overline{r}}(y+l_{\overline{r}}) &= \Omega_{\overline{r}}^0\, U_{\overline{r}}(y)\, \Omega_{\overline{r}}^{-1} \,. && (4.41) \end{aligned}$$

To simplify the notation in what follows, let us define

$$\begin{aligned} s &\equiv \{s_1, s_2, ..., s_{\overline{r}-1}\} \equiv \{y_1, .., y_{\overline{r}-1}\} \,, \\ t &\equiv y_{\overline{r}} \,, \\ u &\equiv \{y_{\overline{r}+1}, .., y_{2n}\} \,, \end{aligned} \tag{4.42}$$

in such a way that $y = \{y_1, y_2, y_3, ..., y_{2n}\} \equiv \{s, t, u\}$. In addition, we denote with $I^{\overline{r}-1}$, the $(\overline{r}-1)$-cube defined as

$$I^{\overline{r}-1} \equiv \{(s_1, ..., s_{\overline{r}-1}) \,|\, 0 \le s_i \le l_i \ \ (1 \le i \le \overline{r}-1)\} \,, \tag{4.43}$$

and by $\partial I^{\overline{r}-1}$ the boundary of $I^{\overline{r}-1}$, defined as

$$\partial I^{\overline{r}-1} \equiv \left\{(s_1, ..., s_{\overline{r}-1}) \in I^{\overline{r}-1} \,|\, \text{some } \ s_i = 0 \text{ or } l_i\right\} \,. \tag{4.44}$$

A possible choice compatible with the periodicity condition in eq. (4.41) is

$$\begin{aligned} U_{\overline{r}}(s,0,u) = & \quad 1 & \equiv \mathcal{C}(s,u) \\ U_{\overline{r}}(s,l_{\overline{r}},u) = & \quad \Omega^0_{\overline{r}}(s,0,u)\,\Omega^{-1}_{\overline{r}}(s,0,u) & \equiv \mathcal{D}(s,u)\,. \end{aligned} \tag{4.45}$$

Using the consistency conditions in eq. (4.34), the periodicity conditions in eq. (4.36) and the constraints in eq. (4.37), it is possible to prove that the choice in eq. (4.45) satisfies the periodicity conditions in eq. (4.40). Furthermore, it is easy to check that for $r < \overline{r}$, it results

$$\mathcal{D}(s+l_r,\,u) \quad = \quad \Omega^0_r\,\mathcal{D}(s,\,u)\,\Omega^{0\;-1}_r = \mathcal{D}(s,\,u)\,. \tag{4.46}$$

$\mathcal{C}(s,u)$ and $\mathcal{D}(s,u)$ are two $(\overline{r}-1)$-loops $\mathcal{C},\mathcal{D} : I^{\overline{r}-1}\times T^{2n-\overline{r}} \to \mathcal{G}$ with base point $g_{\mathcal{C}}$, $g_{\mathcal{D}} \in \mathcal{G}$ respectively. They map, indeed, all points of the boundary $\partial I^{\overline{r}-1}$ into $g_{\mathcal{C}}$, $g_{\mathcal{D}} \in \mathcal{G}$ respectively:

$$\begin{aligned} \mathcal{C}(s|_{\partial I^{\overline{r}-1}},u) &= g_{\mathcal{C}} = 1\;, \\ \mathcal{D}(s|_{\partial I^{\overline{r}-1}},u) &= g_{\mathcal{D}} = \Omega^0_{\overline{r}}(0,0,u)\,\Omega^{-1}_{\overline{r}}(0,0,u)\,. \end{aligned} \tag{4.47}$$

To determine the existence of a gauge transformation $U_{\overline{r}}(y) \in \mathcal{G}$ satisfying eq. (4.45), is therefore tantamount to verify that the $(\overline{r}-1)-$loops $\mathcal{C}(s,u)$ and $\mathcal{D}(s,u)$ are homotopic.

Suppose, now, that for any Ω_a and Ω^0_a, all $U_r(y)$ with $r < \overline{r}$ exist. This implies that all $(r-1)$-loops are already homotopic. Thus, the gauge group $\mathcal{G}$, seen as a topological space, does not contain r-dimensional holes with $r < \overline{r}$.

Under these conditions, two $(\overline{r}-1)-$loops $\mathcal{C}(s,u)$ and $\mathcal{D}(s,u)$ would be homotopic regardless of Ω_a and Ω^0_a, if and only if the $(\overline{r}-1)$-th homotopic group of $\mathcal{G}$ is trivial. The existence of the transformation $U_{\overline{r}}(y)$ guarantees, therefore, that $\overline{r}$-dimensional holes do not exist.

The existence of $U(y)$ defined in eq. (4.38) $\forall\,\Omega^0_a$ and Ω_a is, therefore, necessary and sufficient for $\mathcal{G}$ to be $(2n-1)$-connected.

Summarizing, we have shown that depending on the gauge group $\mathcal{G}$ and on the number of dimensions of the torus, eq. (4.32) may admit solution independently on the choice of Ω_a and V_a, satisfying eq. (4.4) and eq. (4.33), respectively. For example, for a $SU(N)$ gauge theory on a two-dimensional torus, since such group is simply connected (that is $\Pi_1(SU(N)) = 0$), two sets of solutions of the 't Hooft consistency condition are always gauge equivalent: eq. (4.32) always admits a solution.

If we increase the number of dimensions of the torus or change the gauge group, this result does not remain necessarily valid. For example:

- $SU(N)$ is not 3−connected, since $\Pi_3\,(SU(N)) = \mathbf{Z}$. In consequence, if we consider $SU(N)$ on T^4 not all the sets of transition functions are gauge equivalent to the constant ones.

- $U(N)$ is not simply connected and then, for $U(N)$ on T^2, there exist solutions of the consistency conditions in eq. (4.34), inequivalent to the constant ones.

4.3 Background *symmetric gauge*: vacuum symmetries and four-dimensional spectrum

Below, we find and catalogue the possible different classical vacua for a $SU(N)$ theory on a T^2, discuss their symmetries and compute the effective four-dimensional spectrum of fluctuations $\{A_\mu, A_a\}$.

Such exercise can turn out to be very complicate since, in a general background gauge, we have at the same time non-trivial transition functions Ω_a and non-trivial vacuum gauge configuration B_a. To simplify the discussion, it is useful to work in the background *symmetric gauge*: the gauge in which $B_a^{sym} = 0$ and $\Omega_a^{sym} = V_a$.

The $SU(N)$ gauge transformation $S(y)$ that allows to go in the symmetric gauge is simply[6] $S(y) = U^\dagger(y)$ where $U(y)$ is defined in eq. (4.32).

Since in the background *symmetric gauge* $B_a^{sym} = 0$, the symmetries of the vacua correspond to the symmetries of the non-trivial transition functions $\Omega_a^{sym} = V_a$. The periodicity conditions for the fluctuations fields $A_M = \{A_\mu, A_a\}$ reduce, in this gauge, to

$$A_M(y + l_a) = V_a\, A_M(y) V_a^\dagger\,, \tag{4.48}$$

and, therefore, the residual symmetries are associated to the $SU(N)$ generators that commute with V_a.

We divide our analysis in two cases:

- Trivial 't Hooft non-abelian flux: $m = 0$.

- Non-trivial 't Hooft non-abelian flux: $m \neq 0$.

4.3.1 Trivial 't Hooft flux: $m = 0$

The transition functions commute and all the classical vacuum configurations are degenerate in energy with the trivial $SU(N)$ symmetric vacuum. The V_a can be parametrized as

$$V_a = e^{2\pi i \alpha_a^j H_j}\,, \tag{4.49}$$

[6] For $SU(N)$ on T^2, the existence of at least one $U(y)$ satisfying eq. (4.32) for any pair of $\Omega_a(y)$ and V_a satisfying respectively the consistency conditions of eq. (4.4) and eq. (4.33) with the same values of the 't Hooft non-abelian flux m, have been shown in section 4.2.

where H_j are the $N-1$ generators of the Cartan subalgebra of $SU(N)$. V_a, and therefore the vacua, are characterized by $2(N-1)$ real continuous parameters α_a^j, $0 \leq \alpha_a^j < 1$. α_a^i are non-integrable phases, which arise only in a topologically non-trivial space and cannot be gauged-away. Their values must be dynamically determined at the quantum level: only at this level the degeneracy among the infinity of classical vacua is removed [60–62].

The solution with $\alpha_a^j = 0$ is the trivial one. For $\alpha_a^i \neq 0$, the residual gauge symmetries are those associated with the generators that commute with V_a. As V_1 and V_2 commute, the symmetry breaking is rank-preserving and the maximal symmetry breaking pattern that can be achieved is $SU(N) \longrightarrow U(1)^{N-1}$.

The spectrum of the fluctuations reflects the symmetry breaking pattern and it is a function of the non-integrable phases. To give an explicit expression of that spectrum, it is useful to use the Cartan-Weyl basis for the $SU(N)$ generators. In addition to the generators of the Cartan subalgebra H_j with $j = 1, .., N-1$ that satisfy

$$[H_{j_1}, H_{j_2}] = 0 \quad \forall j_1, j_2 = 1, ..., N-1\,, \tag{4.50}$$

we denote as E_r, $r = 1, ..., N^2 - N$, all other $SU(N)$ generators such that

$$[H_j, E_r] = q_r^j E_r \quad \forall\, j = 1, ..., N-1 \ \text{ and } \ \forall\, r = 1, ..., N^2 - N\,. \tag{4.51}$$

In this basis, the four-dimensional mass spectrum for a gauge field A_M^j belonging to the Cartan subalgebra, is the ordinary Kaluza-Klein (KK) spectrum

$$m_{(j)}^2 = 4\pi^2 \left[\frac{n_1^2}{l_1^2} + \frac{n_2^2}{l_2^2}\right]\,, \qquad\qquad n_1, n_2 \ \in \mathbf{Z}\,. \tag{4.52}$$

For a gauge field A_M^r associated to the generator E_r, the mass spectrum reads

$$m_{(r)}^2 = 4\pi^2 \left[\left(n_1 + \sum_{j=1}^{N-1} q_r^j \alpha_1^j\right)^2 \frac{1}{l_1^2} + \left(n_2 + \sum_{j=1}^{N-1} q_r^j \alpha_2^j\right)^2 \frac{1}{l_2^2}\right]\,. \tag{4.53}$$

For all $\alpha_a^j \neq 0$, the only four-dimensional gauge fields that continue to be massless are the $N-1$ fields belonging to the Cartan subalgebra: the spectrum shows the expected maximal symmetry breaking pattern: $SU(N) \rightarrow U(1)^{N-1}$. Finally, notice that the spectra described by eqs.(4.52)-(4.53) depend on the gauge indices but do not depend on the Lorentz ones: from the four-dimensional point of view, the scalars and the gauge bosons coming from internal and ordinary components of a higher-dimensional gauge field, respectively, are expected to be degenerate.

4.3.2 Non-trivial 't Hooft flux: $m \neq 0$

In this case, the transition functions do not commute and all stable vacuum configurations induce some symmetry breaking. For $m \neq 0$, eq. (4.33) reduces to the so-called two-dimensional twist algebra [96]. The possible solutions are of the type [70]

$$\left\{ \begin{array}{lcl} V_1 & = & P^{\alpha_1} Q^{\beta_1} \\ V_2 & = & P^{\alpha_2} Q^{\beta_2} \end{array} \right. , \tag{4.54}$$

where the constant $N \times N$ matrices P and Q are defined as

$$\left\{ \begin{array}{lcl} (P)_{kj} & = & e^{-2\pi i \frac{(k-1)}{N}} \, e^{i\pi \frac{N-1}{N}} \, \delta_{kj} \\ (Q)_{kj} & = & e^{i\pi \frac{N-1}{N}} \, \delta_{k,j-1} \end{array} \right. , \tag{4.55}$$

and satisfy the conditions $P^N = Q^N = e^{\pi i (N-1)}$ and $PQ = e^{\frac{2\pi i}{N}} QP$. While for $m = 0$ we have a continuum of classical vacua characterized by the $2(N-1)$ continuous parameters α_a^j, for $m \neq 0$ we have a finite number of classical vacua characterized by discrete parameters α_1, α_2, β_1, $\beta_2 \in [-N+1, N-1]$, which have to satisfy the consistency condition

$$\alpha_1 \beta_2 \; - \; \alpha_2 \beta_1 \; = \; m \, . \tag{4.56}$$

Notice that α_1, α_2, β_1 and β_2 cannot be simultaneously zero. We introduce the following basis for the generators of $SU(N)$:

$$\tau(\Delta, k_\Delta) \;\; = \;\; \sum_{n=1}^{N} e^{2\pi i \frac{n}{N} k_\Delta} \, \lambda_{(n,n+\Delta)} \, , \tag{4.57}$$

where $\Delta = 0, 1, ..., N-1$, and $k_{\Delta=0} \equiv k_0 = 1, ..., N-1$ and $k_{\Delta \neq 0} = 0, 1, ..., N-1$. The matrices $\lambda_{(n,m)}$ are the $N \times N$ matrices defined by $\left(\lambda_{(n,m)}\right)_{ij} \equiv \delta_{ni}\delta_{mj}$. The traceless matrices $\tau(\Delta, k_\Delta)$ are eigenstates of the operators P, Q with eigenvalues $e^{2\pi i \frac{\Delta}{N}}$ and $e^{2\pi i \frac{k_\Delta}{N}}$, respectively.

The symmetries of the each vacuum are associated to the $SU(N)$ generators that commute simultaneously with V_1 and V_2, that is those $\tau(\Delta, k_\Delta)$ satisfying

$$V_a \, \tau(\Delta, k_\Delta) \, V_a^\dagger \;\; = \;\; e^{2\pi i \frac{\alpha_a \Delta \, + \beta_a k_\Delta}{N}} \, \tau(\Delta, k_\Delta) \qquad \text{for} \quad a = 1, 2 \, . \tag{4.58}$$

with

$$\frac{\alpha_a \Delta \; + \beta_a k_\Delta}{N} \;\; \in \;\; \mathbf{Z} \, . \tag{4.59}$$

We denote[7]

$$\mathcal{K}_1 \;\; = \;\; g.c.d.\,(m, N) \; , \qquad \mathcal{K}_2 = g.c.d.(\alpha_1, \, \alpha_2, \, \beta_1, \, \beta_2, N) \, . \tag{4.60}$$

[7]g.c.d.= great common divisor.

Using eq. (4.56), it is possible to prove that $\mathcal{K}_2 \leq \mathcal{K}_1$ and $\mathcal{K}_1/\mathcal{K}_2 \in \mathbf{Z}$, see Appendix A. The number of $SU(N)$ generators $\tau\,(\Delta, k_\Delta)$ satisfying the condition in eq. (4.59), that is the dimension of the residual symmetry group, Dim$[\mathcal{H}]$ with $\mathcal{H} \subseteq SU(N)$, can be expressed in terms of these two parameters as follows[8]

$$\mathrm{Dim}\,[\mathcal{H}] \;=\; \mathcal{K}_1\,\mathcal{K}_2 - 1\,. \tag{4.61}$$

The corresponding symmetry breaking pattern can be summarized as

$$SU(N) \rightarrow SU(\mathcal{K}_2)^{\frac{\mathcal{K}_1}{\mathcal{K}_2}} \times U(1)^{\frac{\mathcal{K}_1}{\mathcal{K}_2}-1}\,. \tag{4.62}$$

For the following special cases, eq. (4.62) implies

$$\begin{array}{ccc} \mathcal{K}_2 = \mathcal{K}_1 = 1 & , & SU(N) \rightarrow \emptyset\,, \\ \mathcal{K}_2 = 1\,,\ \ \mathcal{K}_1 > 1 & , & SU(N) \rightarrow U(1)^{\mathcal{K}_1 - 1}\,, \\ \mathcal{K}_2 = \mathcal{K}_1 > 1 & , & SU(N) \rightarrow SU(\mathcal{K}_2)\,. \end{array} \tag{4.63}$$

Notice that, given m and N (and consequently $\mathcal{K}_1$), it is possible to have different degenerate vacua characterized by different sets of discrete parameters α_1, α_2, β_1, β_2. They correspond to different values of $\mathcal{K}_2$ and therefore different residual symmetries. Only quantum effects remove such degeneration and determine the true vacuum of the theory. Notice that our result is quite different to the one present in literature [102, 105].

The effective four-dimensional mass spectrum is, also in this case, independent of the Lorentz index $M = 0, 1, ..., 5$, and takes the following form

$$m^2_{(\Delta,k_\Delta)} \;=\; 4\pi^2 \sum_{i=1}^{2} \left(n_i + \frac{\alpha_i\,\Delta \;+\; \beta_i\,k_\Delta}{N}\right)^2 \frac{1}{l_i^2} \qquad n_1, n_2 \in \mathbf{Z}\,. \tag{4.64}$$

The spectrum reflects the symmetry breaking pattern discussed before: given α_1, α_2, β_1, β_2, there exist a zero mode for each gauge boson associated to the generators $\tau(\Delta, k_\Delta)$, with Δ and k_Δ satisfying the condition in eq. (4.59). Since α_1, α_2, β_1, β_2 cannot be simultaneously zero, the spectrum described by eq. (4.64) exhibits always some degree of symmetry breaking.

Notice, also, that the spectra for the case $m = 0$ and $m \neq 0$, in eq. (4.53) and eq. (4.64) respectively, show a similar structure with the only difference that the symmetry breaking contribution to the masses are expressed in terms of continuous ($m = 0$) and discrete ($m \neq 0$) parameters. While in the $m = 0$ case the scale of the lightest non-zero masses $2\pi\alpha_a/l_a$, $a = 1, 2$, is arbitrary and it is fixed only at the quantum level, for the $m \neq 0$ case the non-trivial constraint in eq. (4.4) determines the new scales $\frac{2\pi}{l_a}\,\frac{1}{N}$, already at the

[8]See Appendix A for the details of demonstration.

classical level.

Finally, it is worth to underline the different nature of the symmetry breaking for the two cases of trivial ($m = 0$) and non-trivial ($m \neq 0$) 't Hooft non-abelian flux. In the case $m = 0$, the gauge symmetry breaking mechanism is, indeed, exactly like the Hosotani mechanism [60–62]: it is always possible to choose an appropriate background gauge, compatible with the consistency conditions, in which the transition functions are trivial ($V_1 = V_2 = \mathbf{1}$) and the extra space-like components of the six-dimensional gauge fields $\mathbf{A}_a$ acquire a vacuum expectation value (VEV): $\langle \mathbf{A}_a \rangle = B_a$. In this case, the symmetry breaking can be seen as spontaneous in the following sense:

1. For each 4-dimensional massive gauge field $\mathbf{A}_\mu$, there exists a linear combination of the $\mathbf{A}_a$ that play the role of a 4-dimensional scalar pseudo-goldstone boson, eaten by the 4-dimensional gauge bosons to become a longitudinal gauge degree of freedom.

2. The VEV of $\mathbf{A}_a$ works as the order parameter of the symmetry breaking mechanism. In particular, it is possible to deform $\langle \mathbf{A}_a \rangle$ to zero compatibly with the consistency conditions, so as to restore all the initial symmetries.

In the case $m \neq 0$, we cannot interpret the symmetry breaking mechanism as a spontaneous symmetry breaking mechanism. The consistency conditions, indeed, forbid to have trivial transition functions and then *the symmetry breaking can not be related* ***only*** *to the VEV of* $\mathbf{A}_a$. Although for each massive 4-dimensional gauge boson $\mathbf{A}_\mu$ there exists a 4-dimensional pseudo-goldstone boson, it is not possible to determine an order parameter that can be deformed compatibly with the consistency conditions in such a way to restore all the initial symmetries.

4.4 Conclusions

We have studied extra-dimensional gauge theories with the extra dimensions compactified *á la* Scherk-Schwarz on toroidal manifolds.

Using the analogy with the harmonic oscillator, we have analyzed the vacuum energy for a general group on an even-dimensional torus. For the particular case of $SU(N)$ on T^2, we have re-obtained the well-known result that all stable vacua compatible with four-dimensional Poincaré invariance and zero four-dimensional instanton number have zero energy.

We have, then, studied the classical zero-energy vacua, for a gauge theory on an even-dimensional torus, with periodicity conditions satisfying the 't Hooft consistency conditions. In $SU(N)$ on T^2 case, for each gauge inequivalent set of *constant* transition functions V_a there exists one degenerate and gauge inequivalent classical zero-energy vacuum. We have explicitly demonstrated that such result depends on the particular choice of the gauge group and of the number of extra dimensions.

The number of vacua, the residual symmetries and the nature of the symmetry breaking mechanism depend on the value of the 't Hooft non-abelian flux:

- For trivial 't Hooft flux, $m = 0$, it results a continuum of vacua, degenerate at the classical level with the $SU(N)$ symmetric one. The symmetry breaking is rank-preserving and spontaneous since it is exactly as the Hosotani mechanism.

- The main novel result of this chapter is the explicit demonstration of the symmetry breaking pattern and the four-dimensional mass spectrum for the case of non-trivial 't Hooft flux. For $m \neq 0$, a finite number of vacua results and $SU(N)$ is broken in all of them. The symmetry breaking is rank-lowering and the 't Hooft consistency conditions forbid to interpret it as a spontaneous symmetry breaking.

Chapter 5

1-loop analysis of generalized Scherk-Schwarz symmetry breaking

We analyze below the quantum stability of the symmetry breaking mechanism in the presence of periodic conditions along non-contractible cycles of a non-simply connected manifold. We focus on a $SU(N)$ gauge theory on a two-torus with either trivial or non-trivial 't Hooft flux.

5.1 Heat kernel and 1-loop effective potential

The heat kernel associated to a field theory defined on a general manifold[1] is a very efficient way of discussing renormalizability and computing counterterms, as well as studying other quantum effects such as vacuum polarization, anomalies and the Casimir effect. The reason is its intimate relation with the one loop effective action, explicitly

$$W \equiv \frac{1}{2} \log \det \, A = -\frac{1}{2} \int_0^\infty \frac{dt}{t} \, G(t) \,, \tag{5.1}$$

where A is the operator representing the quadratic part of the action (usually after having expanded around an arbitrary background field), while $G(t)$ is the kernel of such an operator. The heat function $G(X, Y, t)$ defined as

$$G(t) = \int d^{4+d}X \, \mathrm{Tr} \, G(X, X, t) \,, \tag{5.2}$$

is forced to satisfy the heat equation

$$A \, G(X, Y, t) = -\frac{\partial}{\partial t} \, G(X, Y, t) \,, \tag{5.3}$$

[1]Here, only the equations corresponding to a flat manifold are considered, but they can be easily extended to curved ones. For an extense review of mathematical formulations and physical applications of the heat kernel technique see for example [110].

subject to the initial condition

$$G(X,Y,t=0) \;=\; \delta^{4+d}(X-Y)\,. \tag{5.4}$$

In terms of the eigenfunctions f_n and the eigenvalues a_n (assumed positive, real and discrete) of the operator A, the heat function takes the form

$$G(X,Y,t) \equiv \sum_n e^{-a_n t}\, f_n(X)\, f_n^*(Y)\,. \tag{5.5}$$

The completeness relation for the eigenfunctions allows to verify that this expression satisfies the initial condition of eq.(5.4).

On the other hand, the effective action for a field theory is in general a divergent quantity and requires regularization. A very elegant way of regularizing is using the ζ-function technique. The generalized ζ-function associated to an operator A is defined by

$$\zeta_A(s) \;=\; \sum_n \frac{1}{a_n^s}\,, \tag{5.6}$$

and is related to the heat kernel by a Mellin transformation

$$\zeta_A(s) \;=\; \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\int d^{4+d}X\;\mathrm{Tr}\;G(X,X,t)\,, \tag{5.7}$$

in such a way that the effective action is simply

$$W = -\frac{1}{2}\zeta_A'(0)\,. \tag{5.8}$$

This expression is not yet finite and requires regularization. This is provided by analytic continuation to [111, 112]

$$W(s) = -\frac{1}{2}\,\mu^{2s}\,\Gamma(s)\,\zeta_A(s)\,, \tag{5.9}$$

where μ is a constant with mass dimension one introduced to keep the effective action dimensionless. The regularization is removed in the limit $s \to 0$. Using analytic properties of both the Γ- and ζ-functions, it can be shown that the ($\overline{\mathrm{MS}}$) renormalized effective action is given by

$$W^{ren} = -\frac{1}{2}\,\zeta_A'(0) - \frac{1}{2}\log\mu^2\zeta_A(0)\,. \tag{5.10}$$

This is the equation that will be mainly used in the next sections, since we will be interested in the one-loop effective potential, which is, up to a volume factor, the effective action for a constant classical field configuration. It has several advantages with respect to other computational methods: it is manifestly gauge-invariant, the calculation is performed in position space and explicitly in the whole manifold.

All the previous reasoning goes through independently of the manifold considered. It is pertinent, thought, to analyze some subtleties for the case of non-simply connected manifolds, before proceeding further.

In this case, indeed, periodicity conditions along non-contractibles cycles have to be specified, and additional non trivial twists T_a can be considered. For instance, if ϕ is a field in the fundamental representation of the considered gauge group, such periodicity conditions read

$$\phi(x, y + 2\pi R_a) = T_a \phi(x, y) T_a^\dagger \,, \tag{5.11}$$

where x and y denote a point on the ordinary four-dimensional space-time and on the $\mathcal{T}^2$, respectively. $2\pi R_a$ parametrizes the length of the cycle a in term of a radius R_a. The non-trivial periodicity should be reflected in the choice of initial condition in eq.(5.4). We make the following ansatz:

$$G\left(\{x_1, y_1\}, \{x_2, y_2\}, 0\right) \,\equiv\, \delta^4(x_1 - x_2)\, \delta^{\text{E.D.}}(y_1, y_2)\,, \tag{5.12}$$

where $\delta^{\text{E.D.}}$ has the desired periodicity. For example, for the orthogonal two-torus case, it takes the form

$$\delta^{\text{E.D.}}(y_1, y_2) \equiv \prod_{a=1}^{2} \left(\sum_{m_a} \delta(y_1^{(a)} - y_2^{(a)} + 2\pi R_a m_a)\, T_a^{m_a} \right) \,, \qquad y_1^{(a)}, y_2^{(a)} \in [0, 2\pi R_a[\;\; . \tag{5.13}$$

In eq.(5.13), $y^{(a)}$ denotes the coordinate of the a-th extra dimension. The integers m_a can be interpreted as winding numbers. Indeed, they take into account how many times one has to wind around the cycle a in order to go from the point with coordinate $y_1^{(a)}$ to the point with coordinate $y_2^{(a)}$. The introduction of twist T_a in the initial condition ensures the desired periodicity of the heat function and therefore of the effective potential, as well as their gauge invariance. In particular, it allows to consider from the beginning the contribution to the 1-loop effective action stemming from non-local operators such as non-contractible Wilson loops. In other words, without the generalization of the initial condition in eq.(5.13), it is not possible to give a gauge invariant formulation of the non-local contributions to the effective action, due to the non-simply connected nature of the compactified manifold.

5.2 $SU(N)$ 1-loop effective potential on $\mathcal{M}_4 \times T^2$

In this section, the heat kernel technique is applied to compute the 1-loop effective action for a $SU(N)$ gauge background[2] $B_M(x, y)$ with the following characteristics:

[2]Recall that $M = 0, 1, ..., 5$ and $a = 1, 2$.

1. It lives only on the two extra dimensions compactified on $\mathcal{T}^2$ and preserves 4-dimensional Poincaré invariance: $B_M(x, y) \equiv (0, B_a(y))$.

2. It is compatible with the following **constant** periodicity conditions around the non-contractible cycles of the torus:

$$B_a(y + 2\pi R_b) \;=\; T_b\, B_a(y) T_b^\dagger \qquad a, b = 1, 2\ . \tag{5.14}$$

3. It is constant and has zero-field strength. As proved in previous chapters, all $SU(N)$ stable configurations on $\mathcal{T}^2$ have zero-field strength and are all (gauge) equivalent to the constant one (up to a re-definition of periodicity conditions).

As discussed in chapters 3 and 4, the constant periodicity conditions T_1 and T_2 have to satisfy 't Hooft consistency conditions for a fixed value of 't Hooft non-abelian flux. The 1-loop analysis should be divided considering separately the different values of 't Hooft non-abelian flux, or in other words analyzing individually each $SU(N)/\mathbb{Z}_N$ on $\mathcal{T}^2$. We develop below a general formalism which automatically includes all different cases.

To do that, let us choose for each $SU(N)$ representation r, a basis[3] in which the twists T_a are diagonal:

$$\left(T_a^{(r)}\right)_{lm} \;=\; e^{2\pi i q^l_{j_r} \alpha_a^{j_r}}\, \delta_{lm} \qquad l, m = 1, .., N\,, \tag{5.15}$$

where the sum over the index j_r is the sum over gauge space. For example, in the case of the adjoint representation, j_r is the sum over the Cartan sub-algebra and over Δ, k_Δ (see chapter 3, 4) for trivial and non-trivial 't Hooft non-abelian flux, respectively. The parameters $\alpha_a^{j_r}$, finally, are generic and only constrained by the 't Hooft consistency conditions. They are continous or discrete, depending on the particular choice of 't Hooft non-abelian flux. With this notation, eq.(5.12) can be written as

$$\begin{aligned} G\left(\{x_1, y_1\}, \{x_2, y_2\}, 0\right) \;&=\; \delta^4(x_1 - x_2)\, \delta^{\text{E.D.}}\,, \\ \delta^{\text{E.D.}} \;&\equiv\; \prod_{a=1}^{2} \left(\sum_{m_a} \delta(y_1^{(a)} - y_2^{(a)} + 2\pi R_a m_a)\, e^{2\pi i q^l_{j_r} \alpha_a^{j_r} m_a} \right)\,, \end{aligned} \tag{5.16}$$

For constant twists, the periodicity conditions in eq.(5.14) imply that constant $SU(N)$ backgrounds B_a have to commute with T_1 and T_2. In addition such backgrounds have zero field strength and thus they satisfy $[B_a, B_b] = 0$. *It is always possible to find, thus,*

[3]Notice that is always possible to find that basis including the case of non-trivial 't Hooft flux. In fact, the only field representations allowed are those insensitive to the center of the group and therefore those representations for which the constant twist T_a commute simultaneously.

a basis for the $SU(N)$ algebra in which background and constant twists are diagonal. In this basis, it results

$$(B_a)_{lm} = \frac{q_j^l \beta_a^j}{gR_a}\,\delta_{lm}\,. \tag{5.17}$$

Also in this case, the sum over the index j is the sum over the gauge indices.

Finally, we choose the following fluctuation gauge fixing term compatible with the constant periodicity conditions in eq.(5.14)

$$D_M A^M = 0\,. \tag{5.18}$$

Armed with the tools described above, we can compute the 1-loop effective action. Recent literature [113, 114] has evidenced that, at 1-loop, the extra-dimensional and the four-dimensional computation of the same quantity do not necessarily coincide. In particular, the counterterms necessary to remove the 1-loop divergences show some differences in the two cases. Such differences are present also when all Kaluza-Klein modes are included in the four-dimensional computation. At least locally, the effective four dimensional theory with all Kaluza-Klein modes is not able to completely reproduce the properties of the extra-dimensional theory. For this reason, in our computation we will adopt both the extra- and the four-dimensional point of view. In such a way, we will evidence that non-local and finite effects can be equivalently described using both points of view.

5.2.1 Extra-dimensional computation

Consider the contribution to the 1-loop effective potential for B_a, due to gauge (vector) fields and ghosts. For gauge fields, the quadratic operator A in eq.(5.1) reduces to

$$A \equiv -\partial_\mu\partial^\mu - D_a D^a = -\partial_\mu\partial^\mu - (\partial_a + igB_a)(\partial^a + igB^a)\,, \tag{5.19}$$

where the sum over the indices μ and a is implicit. The first step is to solve the heat equation in eq.(5.3) with initial condition given by eq.(5.16). The solution, as can be seen by inspection, is given by

$$G\,(p_1, p_2, t) = \frac{1}{(4\pi t)^3}\sum_{m_1,m_2} e^{-\frac{1}{4t}\left((x_1-x_2)^2+\sum_{a=1}^2(y_1^{(a)}-y_2^{(a)}+2\pi R_a m_a)^2\right)}\; W_1^{m_1}(y_1,y_2)\,W_2^{m_2}(y_1,y_2)\,, \tag{5.20}$$

where $p_1 \equiv \left\{x_1, y_1^{(1)}, y_1^{(2)}\right\}$, $p_1 \equiv \left\{x_1, y_1^{(1)}, y_1^{(2)}\right\}$ and $y_1^{(a)}, y_2^{(a)} \in [0, 2\pi R_a[$. The overall constant factor in eq.(5.20) has been fixed using the definition of the Dirac delta:

$$\delta(x) \equiv \lim_{\epsilon^+ \to 0} \frac{1}{\sqrt{4\pi\epsilon}}\; e^{-\frac{x^2}{4\epsilon}}\,. \tag{5.21}$$

$W_a(y_1, y_2)$ is the Wilson line wrapping once the torus non-contractible cycle a. The covariant expression of $W_a(y_1, y_2)$ is given by

$$W_a(y_1, y_2) \;=\; \mathcal{P}\, exp \left\{ -ig \int_{y_2^{(a)}}^{y_1^{(a)}+2\pi R_a} B_a\, dy'_a \right\} T_a \,, \tag{5.22}$$

where $\mathcal{P}$ denotes the usual path-ordering. Considering a path which starts and ends in a given point y, the Wilson loop is obtained:

$$W_a(y, y) \;=\; \mathcal{P}\, exp \left\{ -ig \int_{y^{(a)}}^{y^{(a)}+2\pi R_a} B_a\, dy'_a \right\} T_a \,. \tag{5.23}$$

Notice that for constant background, the Wilson loop does not depend on the particular point y: $W_a \equiv W_a(y, y)$.

The possibility of finding an analytical solution of the heat equation is strongly related to the particular background that we are considering: a constant and zero field strength background. In this case, it is possible to consider a heat function of the type

$$\sum_{m_1, m_2 = -\infty}^{\infty} \mathcal{G}_0^{(m_1,m_2)}(p_1, p_2, t)\;\; \mathcal{G}_1^{(m_1,m_2)}(p_1, p_2)\;\; \mathcal{G}_2^{(m_1,m_2)}(p_1, p_2) \,, \tag{5.24}$$

where at (m_1, m_2) fixed, $\mathcal{G}_0^{(m_1,m_2)}(p_1, p_2, t)$, $\mathcal{G}_1^{(m_1,m_2)}(p_1, p_2)$ and $\mathcal{G}_2^{(m_1,m_2)}(p_1, p_2)$ are three commuting functions which satisfy

$$\begin{aligned}
(-\partial_\mu \partial^\mu - \partial_1 \partial^1 - \partial_2 \partial^2)\; \mathcal{G}_0^{(m_1,m_2)}(p_1, p_2, t) \;&=\; -\frac{\partial}{\partial t} \mathcal{G}_0^{(m_1,m_2)}(p_1, p_2, t) \\
\mathcal{G}_0^{(m_1,m_2)}(p_1, p_2, t = 0) \;&=\; \delta^4(x_1 - x_2) \prod_{a=1}^{2} \delta(y_1^{(a)} - y_2^{(a)} + 2\pi R_a m_a) \,,
\end{aligned} \tag{5.25}$$

and

$$\begin{aligned}
&D_1 \mathcal{G}_1^{(m_1,m_2)}(p_1, p_2, t) = 0 \\
&D_2 \mathcal{G}_2^{(m_1,m_2)}(p_1, p_2, t) = 0 \,, \\
&\mathcal{G}_a^{(m_1,m_2)}(p_1, p_2, t = 0)\; \delta(y_1^{(a)} - y_2^{(a)} + 2\pi R_a m_a) = T^{m_a} \,.
\end{aligned} \tag{5.26}$$

This system only has solution when $[D_1, D_2] = -igF_{12} = 0$. Whereas the solution of the system in eq.(5.25) is the typical heat function for a non-interacting theory given by

$$\mathcal{G}_0^{(m_1,m_2)}(p_1, p_2, t) \;=\; \frac{1}{(4\pi t)^3} e^{-\frac{1}{4t}\left[(x_1 - x_2)^2 + \sum_{a=1}^{2} (y_1^{(a)} - y_2^{(a)} + 2\pi R_a m_a)^2\right]} \,, \tag{5.27}$$

the system in eq.(5.26) coincides with the definition of the Wilson line: the gauge transformation which sets the background to zero. The solutions are of the type

$$\mathcal{G}_a^{(m_1,m_2)}(p_1,p_2,t) = W_a^{m_a}(y_1,y_2) \;=\; \mathcal{P}\,exp\left\{-ig\int_{y_2^{(a)}}^{y_1^{(a)}+2\pi R_a m_a} B_a\,dy'_a\right\} T_a^{m_a}\,. \quad (5.28)$$

The total contribution to the heat function due to gauge (vector) fields and ghosts reads therefore

$$G^{v+gh}(p_1,p_2,t) \;=\; 4\,\mathrm{Tr}\,\left[\,G(p_1,p_2,t)\,\right]\,. \quad (5.29)$$

The overall factor 4 is due to the fact that for a flat manifold and gauge background with zero-field-strength, the only effect of the ghosts is to reduce to 4 the possible polarizations of a 6-dimensional gauge boson[4].

The following step is to compute the ζ-function as in eq.(5.7), one obtains

$$\begin{aligned}
\zeta_A^R(s) &= \frac{4}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}\int d^4x\int_{\mathcal{T}^2} d^2y\;\mathrm{Tr}\;G(p,p,t)\\
&= 4\,\frac{V^{4+2}}{\Gamma(s)}\int_0^\infty dt\,\frac{t^{s-4}}{(4\pi)^3}\sum_{m_1,m_2} e^{-\frac{1}{4t}\sum_{a=1}^2(2\pi R_a m_a)^2}\;\mathrm{Tr}\;(\,W_1^{m_1}\,W_2^{m_2}) \qquad (5.30)\\
&= \frac{4\,V^{4+2}}{(4\pi)^3\Gamma(s)}\left[\left.\frac{t^{s-3}}{s-4}\right|_{t=0}^{t=\infty} + \sum_{m_1,m_2\neq 0}\mathrm{Tr}\;(\,W_1^{m_1}\,W_2^{m_2}\,)\int_0^\infty dt\,t^{s-4}e^{-\frac{1}{4t}\sum_{a=1}^2(2\pi R_a m_a)^2}\right]
\end{aligned}$$

where we have used the fact that, for constant background, the Wilson loop does not depend on the particular point of the torus. V^{4+2} is the product of the 4-dimensional volume and torus area.

When m_1 or/and m_2 is different from zero, both the integral and the sum converge and they can be interchanged. This contribution is expressed in terms of Wilson loops winding the torus non-contractible cycles exclusively. The divergent contribution to $\zeta_A^R(s)$, instead, is related to the $m_1 = m_2 = 0$ case, that is to zero winding numbers. The latter corresponds to local operators contributions and it manifestly does not depend on the background field. This is a consequence of the fact that all $SU(N)$ invariant local operators, indeed, are expressed in terms of powers of the background field strength: working with zero field strength background, it is not possible to generate any local radiative corrections.

[4]The general quadratic fluctuation operators for gauge bosons and ghosts are

$$\begin{aligned}
\text{gauge} &\;\rightarrow\; g_{\mu\nu}D^2 + R_{\mu\nu} - 2igF_{\mu\nu}\,,\\
\text{ghosts} &\;\rightarrow\; D^2\,.
\end{aligned}$$

The finite contribution reads

$$\zeta_A^R(s) = 4\,\frac{V^{4+2}}{\pi^3}\,\frac{\Gamma(4-s)}{4^s\Gamma(s)} \sum_{m_1,m_2\neq 0} \left[(2\pi R_1 m_1)^2 + (2\pi R_2 m_2)^2\right]^{s-3} \mathrm{Tr}\,(\,W_1^{m_1}\,W_2^{m_2}\,)\;. \quad (5.31)$$

Using eq.(5.10), it is finally possible to compute the contribution to the effective action due to gauge bosons and ghosts:

$$W_{eff}^{\mathrm{ren}} \;=\; -4\,\frac{V^{4+2}}{\pi^3} \sum_{m_1,m_2\neq 0} \left[(2\pi R_1 m_1)^2 + (2\pi R_2 m_2)^2\right]^{-3}\; \mathrm{Tr}\,(\,W_1^{m_1}\,W_2^{m_2}\,)\;. \quad (5.32)$$

The contribution to the 1-loop effective action due to N_f fermions and N_s scalars in the representation r of $SU(N)$ can be easily computed in the same way and reads

$$W_{eff}^{\mathrm{ren}\,f} \;=\; 2\,N_f\,\frac{V^{4+2}}{\pi^3} \sum_{m_1,m_2\neq 0} \left[(2\pi R_1 m_1)^2 + (2\pi R_2 m_2)^2\right]^{-3} \mathrm{Tr}_r\,(W_1^{m_1}\,W_2^{m_2}\,)\;, \quad (5.33)$$

$$W_{eff}^{\mathrm{ren}\,s} \;=\; -2\,N_s\,\frac{V^{4+2}}{\pi^3} \sum_{m_1,m_2\neq 0} \left[(2\pi R_1 m_1)^2 + (2\pi R_2 m_2)^2\right]^{-3} \mathrm{Tr}_r\,(\,W_1^{m_1}\,W_2^{m_2}\,)\;. \quad (5.34)$$

where and Tr_r means the trace over the representation r of $SU(N)$.

5.2.2 4-dimensional computation

We aim to reproduce eq.(5.32), starting from the 4-dimensional effective theory, obtained integrating over the two extra dimensions. Let us concentrate only on the 1-loop contribution due to 4-dimensional gauge bosons, ghosts and 4-dimensional scalars $\phi^{(a)} = B_a$, associated to the extra components of a higher-dimensional gauge field.

For gauge and scalar fields with given Kaluza-Klein (KK) (n_1, n_2) and gauge (j) index, the quadratic operator A in eq.(5.3) reduces to

$$A \;\equiv\; -\partial_\mu\partial^\mu - M^2_{(j,n_1,n_2)}\,, \quad (5.35)$$

where

$$M^2_{(j,n_1,n_2)} \equiv \sum_{a=1}^{2} \frac{1}{R_a^2}\,\left(n + q_j^l(\alpha_a^j - \beta_a^j)\right)^2\;. \quad (5.36)$$

The 4-dimensional effective squared mass $M^2_{(j,n_1,n_2)}$ has been computed using the explicit form of twist and constant background in eq.(5.15) and (5.17), respectively. The heat equation in eq.(5.3), with initial condition given by

$$G(x_1, x_2, t=0) = \delta^4(x_1 - x_2)\,, \quad (5.37)$$

admits the following solution:

$$G^{l,n_1,n_2}(x_1,x_2,t) \;=\; \frac{1}{(4\pi t)^2} e^{\frac{-(x_1-x_2)^2}{4t}}\, e^{-M^2_{(j,n_1,n_2)}t}\,. \qquad (5.38)$$

Notice that the last exponential in eq.(5.38) is simply the Fourier transformation of the extra-dimensional contribution in eq.(5.20) after substituting the results in eqs.(5.15) and (5.17).

The indices l, n_1, n_2 remind that the quantity in eq.(5.38) is the contribution due to a degree of freedom which has KK indices n_1, n_2 and it is eigenstate of the periodicity conditions, with eigenvalue $e^{2\pi i q^l_j \alpha^j_a}$. The total heat function due to all 4-dimensional gauge bosons and ghosts reads therefore

$$G^{v+gh}(x_1,x_2,t) \;=\; 2\sum_l \sum_{n_1,n_2} G^{l,n_1,n_2}(x_1,x_2,t)\,, \qquad (5.39)$$

whereas the contribution coming form the scalars is

$$G^{scal}(x_1,x_2,t) \;=\; 2\sum_l \sum_{n_1,n_2} G^{l,n_1,n_2}(x_1,x_2,t)\,. \qquad (5.40)$$

As before, the only effect due to the ghosts is to reduce to 2 the possible polarizations of a 4-dimensional gauge vector field. The overall factor 2 in the scalar contribution takes into account the number of independent 4-dimensional scalars. The sum over l in both cases can be interpreted as the trace over the gauge indices.

The next step is to compute the ζ-function as in eq.(5.7).

$$\begin{aligned}
\zeta^R_A(s) \;&=\; 4\sum_l \sum_{n_1,n_2} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \int d^4x\, G(x,x,t) && (5.41)\\
&=\; 4\sum_l \sum_{n_1,n_2} \frac{V^4}{\Gamma(s)} \int_0^\infty dt\, \frac{t^{s-3}}{(4\pi)^2} e^{-M^2_{(j,n_1,n_2)}t} \\
&=\; \frac{4\,V^4}{(4\pi)^2} \sum_l \sum_{n_1,n_2} (M^2_{(j,n_1,n_2)})^{2-s} \frac{\Gamma(s-2)}{\Gamma(s)}\,, && (5.42)
\end{aligned}$$

Using the formula in eq.(5.10), the effective action reads

$$W^{\rm ren}_{eff} \;=\; -2\,\frac{V^4}{(4\pi)^2} \sum_{n_1,n_2} \sum_l (M^2_{(j,n_1,n_2)})^2 \left(\frac{3}{4} - \frac{1}{2} \log \frac{M^2_{(j,n_1,n_2)}}{\mu^2} \right)\,. \qquad (5.43)$$

It could seems that, in the 4-dimensional computation, operators that do not satisfy the symmetries of the original 6-dimensional theory could be induced. This impression is an artifact of the particular way in which eq.(5.43) is expressed. To clarify this statement, we evaluate the following two series:

$$
\begin{aligned}
1)\quad \sum_{n_1,n_2}\left(\sum_{a=1}^{2}\left(n_a+q_j^l(\alpha_a^j-\beta_a^j)\right)^2\frac{1}{R_a^2}\right)^2 &= \sum_{n_1,n_2}\frac{\partial^2}{\partial\xi^2}e^{-\sum_{a=1}^{2}\frac{\xi}{R_a^2}\left(n_a+q_j^l(\alpha_a^j-\beta_a^j)\right)^2}\Bigg|_{\xi=0}\\
&= \frac{\partial^2}{\partial\xi^2}\prod_{a=1}^{2}\left(\sum_{n_a}e^{-\frac{\xi}{R_a^2}\left(n_a+q_j^l(\alpha_a^j-\beta_a^j)\right)^2}\right)\Bigg|_{\xi=0}\\
&= V^2\,\frac{\partial^2}{\partial\xi^2}\frac{1}{(4\pi\xi)}\left(\sum_{m_1,m_2}e^{-\sum_{a=1}^{2}\frac{(2\pi R_a m_a)^2}{4\xi}}e^{2\pi i q_j^l\sum_{a=1}^{2}(\alpha_a^j-\beta_a^j)m_a}\right)\Bigg|_{\xi=0}\\
&= 2\frac{V^2}{(4\pi)}\sum_{m_1,m_2}\delta_{m_1,0}\,\delta_{m_2,0}\,\frac{1}{\xi^3}\Bigg|_{\xi=0}\,e^{2\pi i q_j^l\sum_{a=1}^{2}(\alpha_a^j-\beta_a^j)m_a}\\
&= 2\frac{V^2}{(4\pi)}\,\frac{1}{\xi^{\frac{4+d}{2}}}\Bigg|_{\xi=0}
\end{aligned}
\tag{5.44}
$$

The result in eq.(5.44) implies that the first contribution to the 4-dimensional effective potential in eq.(5.43) is independent of the background field and it gives rise to a divergence proportional to the volume.

$$
\begin{aligned}
&2)\quad \sum_{n_1,n_2}\left(\sum_{a=1}^{2}\left(n_a+q_j^l(\alpha_a^j-\beta_a^j)\right)^2\frac{1}{R_a^2}\right)^2\log\sum_{a=1}^{2}\frac{\left(n_a+q_j^l(\alpha_a^j-\beta_a^j)\right)^2}{R_a^2\mu^2}=\\
&= -\mu^4\int_0^\infty\frac{dt}{t}\frac{\partial^2}{\partial t^2}\prod_{a=1}^{2}\left(\sum_{n_a}e^{-\frac{\left(n_a+q_j^l(\alpha_a^j-\beta_a^j)\right)^2}{R_a^2\mu^2}t}\right)\\
&= -\int_0^\infty\frac{dt}{t}\frac{\partial^2}{\partial t^2}\prod_{a=1}^{2}\left(\frac{2\pi R_a}{(4\pi t)^{\frac{1}{2}}}\sum_{m_a}e^{-\frac{(2\pi R_a m_a)^2}{4t}}\,e^{2\pi i q_j^l(\alpha_a^j-\beta_a^j)m_a}\right)\\
&= -\frac{V^2}{(4\pi)}\int_0^\infty\frac{dt}{t}\frac{\partial^2}{\partial t^2}\frac{1}{t}\sum_{\vec{m}}e^{-\sum_{a=1}^{2}\frac{(2\pi R_a m_a)^2}{4t}}\,e^{2\pi i q_j^l\sum_{a=1}^{2}(\alpha_a^j-\beta_a^j)m_a}\\
&= -\frac{V^2}{(4\pi)}\left[2\int_0^\infty\frac{dt}{t^3}+\sum_{m_1,m_2\neq 0}e^{2\pi i q_j^l\sum_{a=1}^{2}(\alpha_a^j-\beta_a^j)m_a}\int_0^\infty\frac{dt}{t}\frac{\partial^2}{\partial t^2}\frac{1}{t}\,e^{-\sum_{a=1}^{2}\frac{(2\pi R_a m_a)^2}{4t}}\right]\\
&= -64\,\frac{V^2}{\pi}\sum_{m_1,m_2\neq 0}W_1^{m_1}\,W_2^{m_2}\left[(2\pi R_1 m_1)^2+(2\pi R_2 m_2)^2\right]^{-3}
\end{aligned}
\tag{5.45}
$$

where, in the first step, we have rescaled the proper-time in such a way as to eliminate the explicit dependence on the arbitrary scale μ. In the last step, we have used the definition of the Wilson loop in terms of diagonal twist and background and we dropped the divergent contribution which is proportional to the volume and independent of the background field.

Using the results in eqs.(5.44)-(5.45) and obviating the background-independent contributions, it results

$$W_{eff}^{\rm ren} = -\frac{4}{\pi^3} V^{4+2} \sum_{m_1, m_2 \neq 0} W_1^{m_1} W_2^{m_2} \left[(2\pi R_1 m_1)^2 + (2\pi R_2 m_2)^2\right]^{-2-\frac{d}{2}} . \qquad (5.46)$$

A comparison with eq.(5.32) shows that both the extra- and the four-dimensional computations give rise to same result as regards the finite part of the effective action.

5.3 Discussion

Some preliminary results on the vacuum analysis of the 1-loop effective action are reported in what follows [115].

The radiative analysis evidences that symmetry breaking mechanisms related to periodicity conditions along non-contractible cycles are insensitive to the local dynamics and therefore they are not affected by the hierarchy problem. Such a result works for both trivial and non-trivial 't Hooft non-abelian flux.

The explicit form of the effective potential depends on the choice of 't Hooft non-abelian flux. More in detail, the Wilson loops are function of continous or discrete parameters depending whether the 't Hooft flux is trivial o non-trivial, respectively.

For trivial 't Hooft flux and for a fixed matter content, the 1-loop effective action is a continous function of $2(N-1)$ continous parameters. It is gauge invariant but it may have a vacuum which does not respect all the symmetries of the original theory resulting in a spontaneous symmetry breaking mechanism [60–62].

A comment is interesting at this level. In a local $SU(N)$ invariant theory, non-linear transformation properties of a gauge boson forbid any mass terms. In other words, in the vacuum, each gauge boson is associated to a flat direction, as a consequence of gauge invariance. In the case of a non-local effective action expressed in terms of non-local Wilson loops as in eq.(5.32), the situation is very different. Gauge bosons, indeed, appear in the effective action only through powers of operators of the type

$$\int_y^{y+2\pi R_a} A_a dy^a \, . \qquad (5.47)$$

In the case of trivial periodicity conditions, all A_a are periodic and this expression is invariant under the non-linear part of a gauge transformation. No shift symmetry able to protect the boson mass survives in this case. In the vacuum, therefore, the original gauge invariance does not imply that gauge bosons are associated to a flat direction. Indeed, also when the vacuum expectation value of the Wilson loop is trivial, finite (and completely symmetric) mass terms for A_a appear [116]. In the case of non-trivial constant periodicity conditions, some gauge bosons A_a are not strictly periodic. In this case there are two strictly correlated consequences:

- The ordinary 4-dimensional components A_μ, coming from the same extra dimensional gauge field that gives rise to A_a, is also not strictly invariant. This implies (as we have discussed in chapter 1) that such 4-dimensional gauge bosons acquire a mass.

- The quantity in eq.(5.47) is not invariant under the non-linear part of gauge transformations. Such residual shift symmetry is a direct evidence that these A_a will play the role of pseudo-Goldstone bosons of the spontaneous breaking of symmetries associated to A_μ.

For non-trivial 't Hooft flux and for a fixed matter content, the 1-loop effective action is a discrete function depending on 4 discrete parameters. It is gauge invariant but all vacua are symmetry breaking. In this case, the effective potential is not a continous function and then the symmetry breaking mechanism does not work as a Higgs mechanism. That is, such symmetry breaking mechanism cannot be interpreted as a spontaneous symmetry breaking mechanism. For example no new mass terms for A_a come from the 1-loop effective action. Such a result confirms the discussion in chapter 4, performed only using periodicity conditions, about the nature of the symmetry breaking in the case of 't Hooft non-abelian flux.

Let us only indicate that the 1-loop corrections break the tree-level degeneracy among the classical vacua. The analysis of the true vacuum of the effective theory as well as the discussion of its symmetries is nowadays under investigation [115] and we do not discuss it here.

Chapter 6

Flavour problem

6.1 Flavour physics and flavour parameters

The Standard Model fermions are arranged in a "three generation" pattern. *Flavor physics* describes interactions that distinguish between them.

Fermions experience two types of interactions: gauge interactions and Yukawa. Within the Standard Model [117, 118](SM), there are twelve gauge bosons, related to the gauge symmetry group

$$G_{\mathrm{SM}} = SU(3)_{\mathrm{C}} \times SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}}\,, \tag{6.1}$$

and a single Higgs scalar, related to the spontaneous symmetry breaking mechanism

$$G_{\mathrm{SM}} \to SU(3)_{\mathrm{C}} \times U(1)_{\mathrm{EM}}\,. \tag{6.2}$$

In the interaction basis, gauge interactions are diagonal (and universal, namely described by a single gauge coupling for each factor in G_{SM}: g_s, g and g'). By definition, the interaction eigenstates have no gauge couplings between fermions of different generations. Mass eigenstates differ from interaction eigenstates. In the mass basis, Yukawa interactions are diagonal (though not universal). The mass eigenstates have, by definition, well-defined masses. The gauge interactions related to spontaneously broken symmetries can, however, be quite complicated in the mass basis. In particular, the $SU(2)_{\mathrm{L}}$ gauge couplings are not diagonal, that is they *mix* fermions of different generations. *Flavor Physics* here refers to fermion masses and mixings.

Flavour changing neutral current processes (FCNC) depend on the flavour parameters. For diagonal Yukawa couplings, FCNC would be absent to all orders in the gauge couplings. Consequently, within the SM FCNC are suppressed by small mixing angles and, in some cases, small quark masses. Furthermore, within the SM FCNC vanish at tree level. Consequently, they are further suppressed by powers of the weak coupling. Many extensions of the SM allow significant new contributions to these processes that modify SM predictions. Therefore, the flavour sector is a very sensitive probe of New Physics.

CP violation is also part of flavour physics. The Kobayashi-Maskawa phase [119] has been well measured in processes regarding K and B physics [120]. At the same time, it does not explain the so called "strong CP-problem" neither it is sufficient to explain the matter-antimatter asymmetry of the universe [121, 122]. Almost any extension of the SM provides new sources of CP violation. Such extensions, therefore, have to be able to reproduce the measured K and B physics and at the same time they could be an interesting scenario for the unsolved problems.

Flavour parameters

Each SM fermion generation is made out of five different representations of the SM gauge group G_{SM} in eq.(6.1):

$$Q_{Li}(3,2)_{+1/6}, \quad u_{Ri}(3,1)_{+2/3}, \quad d_{Ri}(3,1)_{-1/3}, \quad L_{Li}(1,2)_{-1/2}, \quad \ell_{Ri}(1,1)_{-1}\,. \tag{6.3}$$

Our notations mean that, for example, the left-handed quarks, Q_L, are in a triplet (3) of the $SU(3)_{\mathrm{C}}$ group, a doublet (2) of $SU(2)_{\mathrm{L}}$ and carry hypercharge $Y = Q_{\mathrm{EM}} - T_3 = +1/6$. The index $i = 1, 2, 3$ is the *flavour* (or generation) index.

The SM gauge interactions do not distinguish between the different generations. Another way to state this is to say that gauge interactions are flavour-blind. The strength of the gauge interactions depends on the gauge quantum numbers given in eq.(6.3) and not on the flavour index i. Most important for our purposes, the interaction of the $SU(2)_L$ gauge bosons (W^a_μ, $a = 1, 2, 3$) with quarks is given by

$$-\mathcal{L}_W = \frac{g}{2}\overline{Q_{Li}}\gamma^\mu \tau^a Q_{Li} W^a_\mu\,. \tag{6.4}$$

The 4×4 matrix γ^μ operates in Lorentz space and the 2×2 matrix τ^a operates in the $SU(2)_{\mathrm{L}}$ space. The coupling $\overline{Q^I_{Li}}Q^I_{Li}$ can be equivalently written as $\overline{Q^I_{Li}}\mathbf{1}_{ij}Q^I_{Lj}$ where the 3×3 unit matrix $\mathbf{1}$ operates in flavour space and makes the universality of the gauge interactions manifest.

The Yukawa interactions in this basis read

$$-\mathcal{L}_Y = Y^d_{ij}\overline{Q_{Li}}Hd_{Rj} + Y^u_{ij}\overline{Q_{Li}}\tilde{H}u_{Rj} + Y^\ell_{ij}\overline{L_{Li}}H\ell_{Rj}\left(+Y^\nu_{ij}\overline{L_{Li}}\tilde{H}N_{Rj} + Mayorana\right)\,, \tag{6.5}$$

where $H(1,2)_{+1/2}$ is the SM Higgs doublet, and $\tilde{H} = i\sigma_2 H^*$. The Yukawa matrices Y^d, Y^u, Y^l and Y^l are general (and, in particular, complex) 3×3 matrices. In what follows, we will concentrate only on the quark sector.

Masses result upon spontaneous symmetry breaking, that is a vacuum expectation value taken by by the neutral component of the Higgs doublet, $\langle\phi^0\rangle = \frac{v}{\sqrt{2}}$. The electroweak breaking scale is fixed from the gauge boson masses and is of order $v \approx 246\ GeV$. Upon the replacement $Re(H^0) \to (v+H^0)/\sqrt{2}$, the Yukawa interactions in eq.(6.5) give rise to mass terms:

$$-\mathcal{L}_M = (M_d)_{ij}\overline{d_{Li}}d_{Rj} + (M_u)_{ij}\overline{u_{Li}}u_{Rj}\,, \tag{6.6}$$

where

$$M_f = \frac{v}{\sqrt{2}}Y^f\,, \tag{6.7}$$

and we have decomposed the $SU(2)_{\rm L}$ doublets in terms of their components:

$$Q_{Li} = \begin{pmatrix} u_{Li} \\ d_{Li} \end{pmatrix}\,, \tag{6.8}$$

The mass basis corresponds, by definition, to diagonal mass matrices. We can always find unitary matrices V_{fL} and V_{fR} such that

$$V_{fL}M_f V_{fR}^\dagger = M_f^{\rm diag}\,, \tag{6.9}$$

with $M_f^{\rm diag}$ diagonal and real. The mass eigenstates are then identified as

$$\begin{aligned} d_{L\alpha} &= (V_{dL})_{\alpha j}d_{Lj} \quad , \quad d_{R\alpha} = (V_{dR})_{\alpha j}d_{Rj}, \\ u_{L\alpha} &= (V_{uL})_{\alpha j}u_{Lj} \quad , \quad u_{R\alpha} = (V_{uR})_{\alpha j}u_{Rj}, \end{aligned} \tag{6.10}$$

Charged current interactions (that is the interactions of the charged $SU(2)_{\rm L}$ gauge bosons $W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2)$), which in the interaction basis are described by eq.(6.4), take the following form in the mass basis:

$$-\mathcal{L}_{W^\pm} = \frac{g}{\sqrt{2}}\overline{u_{L\alpha}}\gamma^\mu(V_{uL}V_{dL}^\dagger)_{\alpha\beta}d_{L\beta}W_\mu^+ + \text{h.c.}\,. \tag{6.11}$$

The 3×3 unitary matrix,

$$V_{\rm CKM} = V_{uL}V_{dL}^\dagger\,, \tag{6.12}$$

is the CKM *mixing matrix* [119,123]. It generally depends on nine parameters: three real angles and six phases, but not all of them are physicals.

The form of the matrix is not unique. Usually, the following two conventions are employed:

(i) There is further freedom in the phase structure of $V_{\rm CKM}$. Let us define P_f ($f = u, d, \ell$) to be diagonal unitary (phase) matrices. Then, if instead of using V_{fL} and V_{fR} for the rotation in eq.(6.10), we use $\tilde{V}_{fL}$ and $\tilde{V}_{fR}$, defined by $\tilde{V}_{fL} = P_f V_{fL}$ and $\tilde{V}_{fR} =$

$P_f V_{fR}$, we still maintain a legitimate mass basis since $M_f^{\rm diag}$ remains unchanged by such transformations. However, $V_{\rm CKM}$ does change:

$$V_{\rm CKM} \to P_u V_{\rm CKM} P_d^* \,. \tag{6.13}$$

This freedom is fixed by demanding that $V_{\rm CKM}$ will have the minimal number of phases. Indeed, what counts is the physical number of phases, that is, the number of phases which cannot be reabsorbed by field redefinitions. In the three generation case $V_{\rm CKM}$ has a single phase. (There are five phase differences between the elements of P_u and P_d and, therefore, five of the six phases in the CKM matrix can be removed.) This is the Kobayashi-Maskawa phase $\delta_{\rm KM}$ which is the single source of *CP violation* in the SM [119].

(*ii*) There is freedom in defining $V_{\rm CKM}$ in that we can permute between the various generations. This freedom is fixed by ordering the up quarks and the down quarks by their masses, i.e. $m_{u_1} < m_{u_2} < m_{u_3}$ and $m_{d_1} < m_{d_2} < m_{d_3}$. (Usually, we call $(u_1, u_2, u_3) \to (u, c, t)$ and $(d_1, d_2, d_3) \to (d, s, b)$.) It is an interesting fact that with this convention $V_{\rm CKM}$ is close to a unit matrix.

As a result of the fact that $V_{\rm CKM}$ is not diagonal, the $W^\pm$ gauge bosons can couple to quark (mass eigenstates) of different generations. Within the SM quark sector, this is the only source of *flavour changing* interactions. Clearly, there are additional sources of flavour mixing in the lepton sector when right-handed neutrinos are included.

We now recall why, within the SM, the Z^0 interactions do not give rise to flavour-changing processes. Defining $\tan\theta_W \equiv g'/g$, it results

$$Z^\mu = \cos\theta_W W_3^\mu - \sin\theta_W B^\mu \,. \tag{6.14}$$

(B is the gauge boson related to $U(1)_{\rm Y}$.) The Lagrangian for W_3-interactions (given in eq.(6.4) and B interactions reads:

$$-\mathcal{L}_B = -g' \left[\frac{1}{6} \overline{Q_{Li}} \gamma^\mu \mathbf{1}_{ij} Q_{Lj} + \frac{2}{3} \overline{u_{Ri}} \gamma^\mu \mathbf{1}_{ij} u_{Rj} - \frac{1}{3} \overline{d_{Ri}} \gamma^\mu \mathbf{1}_{ij} d_{Rj} \right] B_\mu \,. \tag{6.15}$$

Let us examine, for example, the Z-interactions with d_L in the mass basis:

$$\begin{aligned} -\mathcal{L}_Z &= \frac{g}{\cos\theta_W} \left(-\frac{1}{2} + \frac{1}{3} \sin^2\theta_W \right) \overline{d_{L\alpha}} \gamma^\mu (V_{dL}^\dagger V_{dL})_{\alpha\beta} d_{L\beta} Z_\mu \\ &= \frac{g}{\cos\theta_W} \left(-\frac{1}{2} + \frac{1}{3} \sin^2\theta_W \right) \overline{d_{L\alpha}} \gamma^\mu d_{L\alpha} Z_\mu . \end{aligned} \tag{6.16}$$

We learn that neutral current interactions remain universal in the mass basis and there are no additional flavour parameters in their description. This situation goes beyond the SM to all models where all left-handed quarks are in $SU(2)_{\rm L}$ doublets and all right-handed ones in singlets. The Z-boson does have flavour changing couplings in models where this is not the case.

Let us now count how many flavour parameters there are in the SM quark sector. In the interaction basis, the flavour parameters come from the two up and down Yukawa matrices. Since each of these is a 3×3 complex matrix, there are in total 18 real and 18 imaginary parameters. Not all of them are, however, physical. If we switch off the Yukawa matrices, there is a global symmetry added to the SM,

$$G_{\text{global}}(Y^f = 0) = U(3)_Q \times U(3)_{\bar{d}} \times U(3)_{\bar{u}} \,. \tag{6.17}$$

An unitary rotation of the three generations for each of the three quark representations in eq.(6.3) would leave the SM Lagrangian invariant. This means that the physics described by a given set of Yukawa matrices (Y^d, Y^u), and the physics described by another set,

$$\tilde{Y}^d = V_Q^\dagger Y^d V_{\bar{d}}, \quad \tilde{Y}^u = V_Q^\dagger Y^u V_{\bar{u}} \,, \tag{6.18}$$

where V are all unitary matrices, is the same. One can use this freedom to remove, at most, 9 real and 18 imaginary parameters (the number of parameters in five 3×3 unitary matrices). However, the fact that the SM with the Yukawa matrices switched on has still a global symmetry

$$G_{\text{global}} = U(1)_B \tag{6.19}$$

means that only 17 imaginary parameters can be removed. We conclude that there are 10 flavour parameters in quark sector: 9 real ones and a single phase.

Examining the mass basis one can easily identify the flavour parameters. We have six quark masses, three mixing angles (the number of real parameters in V_{CKM}) and the single phase δ_{KM} mentioned above.

6.2 Abelian flavour symmetries: Froggat-Nielsen mechanism

In the SM, the quark masses and the entries of the V_{CKM} matrix are renormalizable parameters fixed by the comparison with the experimental data. The SM, therefore, is not able to explain the special structure of the mass matrices or the consequent mass ratios or the mixing angles present in nature.

In the literature, the general approach used to try to solve such problem, it is to assume the existence of a new symmetry that forbids the existence of some quark (or lepton) mass couplings. In this context the physical masses (finite and experimentally observable) stem in consequence of the breaking of the symmetry in question. These models are developed in different contexts: simple extensions of the SM [6], MSSM [3, 7], supersymmetric GUT [4, 5], extra dimensions [124–126].

Although it is possible consider flavour symmetries with very different characteristics (discrete or continous, local or global, abelian or non-abelian), in this section, we will concentrate on the original idea proposed by C. D. Froggatt and H. B. Nielsen [6] of a global continous abelian flavour symmetry that will be useful for the discussion of chapter 7.

Froggatt-Nielsen mechanism

The simplest realization, consists in enlarging the (gauge and/or global) symmetry group of our theory, introducing an almost conserved abelian global symmetry, $U(1)_{FN}$. The corresponding abelian charge q_{FN} has to be quantized and anomaly free in such a way to be able to predict perturbatively the SM masses.

It is also assumed the existence of additional very heavy fermions Ψ_{FN} taking different flavour charges. The natural mass scale is thus determined by these heavy particles, which is much higher than the electroweak scale.

In the symmetric limit, all presently observed quarks and leptons are effectively massless and we are only observing the low-energy tail of some more fundamental physics.

In the limit of exact q_{FN} conservation, indeed, the usual SM Yukawa couplings are forbidden. In particular, it is possible to realize this through an appropriate assignment of the SM flavour charges in such a way that (some) Yukawa couplings turn out to be non-invariant under the $U(1)_{FN}$ transformations. One possibility would be to give different FN charges q_{FN} to left- and right-handed components of the SM fermions and to set to zero the FN charge of the SM Higgs H. In the original Lagrangian, therefore, terms like

$$Y_d \bar{Q}_L H d_R \qquad\qquad Y_u \bar{Q}_L H^C u_R \,, \tag{6.20}$$

are not $U(1)_{FN}$ invariant and then cannot be written.

In order to generate low energy effective masses for the SM fermions, the matter content of the model has to be enlarged introducing some new SM singlet scalars. Since the new scalar fields are singlets under the SM gauge group, it is not possible to directly couple them to the SM fermions, that is, they cannot be used to generate mass terms like $\bar{Q}_L \phi d_R$. They are, however, responsible of the heavy fermion masses through a Higgs-like mechanism. To be concrete, we assume that there are two new scalars ϕ_0 and ϕ_1, with zero and non-zero FN charge respectively. While ϕ_0 generates (vectorial and $U(1)_{FN}$ preserving) heavy fermion masses of the type

$$\bar{\Psi}_{FN} \, \langle \phi_0 \rangle \, \Psi_{FN} \,, \tag{6.21}$$

the vacuum expectation value $\langle \phi_1 \rangle$ is responsible for the flavour symmetry breaking and consequently for the SM (light) masses.

The $U(1)_{FN}$ symmetry allows interactions between heavy and SM fermions of the type

$$\bar{\Psi}_{FN} \, \phi_1 \, d_R \,, \tag{6.22}$$

and, therefore, the "ordinary" SM masses can be interpreted as a low-energy effect of diagrams of the type in figure 6.1. The double line represent the heavy fermion propagator while H is the SM Higgs which breaks the electroweak symmetry and generates the gauge boson masses.

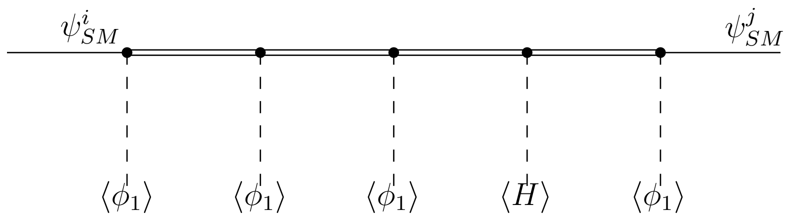

Figure 6.1: Origin of SM masses.

Integrating out the heavy fermions, we get the following effective local interactions

$$\bar{Q}^i_L \, \langle H \rangle \, (Y_d)_{ij} \, \epsilon^{n_{ij}} \, d^j_R \qquad \text{with} \qquad \epsilon \equiv \frac{\langle \phi_1 \rangle}{\langle \phi_0 \rangle} \, , \tag{6.23}$$

and $n_{ij} = q_{FN}(\bar{Q}^i_L) - q_{FN}(d^j_R)$. The effective Yukawa couplings are, therefore, given by

$$\left(Y^{SM}_r\right)_{ij} \; = \; (Y_r)_{ij} \, \epsilon^{n_{ij}} \, , \tag{6.24}$$

where $r = u, d, l, \nu$. In this type of scenario, it is thus possible to note:

1. The flavour symmetry breaking parameter ϵ does not depend on the absolute scale of the new physics. It only depends on the ratio between the two scales $\langle \phi_1 \rangle$ and $\langle \phi_0 \rangle$, which can be a priori arbitrarily big.

2. Including the case in which all tree level Yukawa couplings of the mother theory are of order $\mathcal{O}(1)$, the effective SM Yukawa couplings $\left(Y^{SM}_r\right)_{ij}$ can be made arbitrarily small. Notice, indeed, that in the case $\langle \phi_1 \rangle < \langle \phi_0 \rangle$, to make $\left(Y^{SM}_r\right)_{ij}$ arbitrarily small, it is sufficient to choose the parameter n_{ij} large enough, but it is not necessary a big hierarchy between $\langle \phi_1 \rangle$ and $\langle \phi_0 \rangle$.

3. Assigning different values of flavour charges to different SM fermions, it is possible to generate a hierarchical flavour structure for SM Yukawa couplings.

A realistic implementation of the Froggatt and Nielsen idea, in general, has to face some complications as for example:

- Difficulty in motivating why the flavour charge appears quantized.
- Problems related to anomaly cancellation.
- Difficulty in reproducing the SM Yukawa coupling and (in general) the consequent need of more than one heavy scalar.
- Flavour changing neutral current and unitarity constraints.

In the next chapter, we will present a possible implementation of the Froggatt-Nielsen idea in the context of gauge-Higgs unification in extra dimensions. In such a framework, some of the above-mentioned problems find a natural explanation.

Chapter 7

Minimal gauge-Higgs unification with a flavour symmetry

We show below that a flavour symmetry à la Froggatt-Nielsen can be naturally incorporated in models with gauge-Higgs unification, by exploiting the heavy fermions that are anyhow needed to realize realistic Yukawa couplings. The case of the minimal five-dimensional model, in which the $SU(2)_L \times U(1)_Y$ electroweak group is enlarged to an $SU(3)_W$ group, and then broken to $U(1)_{\rm em}$ by the combination of an orbifold projection and a Scherk-Schwarz twist, is studied in detail. We show that the minimal way of incorporating a $U(1)_F$ flavour symmetry is to enlarge it to an $SU(2)_F$ group, which is then completely broken by the same orbifold projection and Scherk-Schwarz twist. The general features of this construction, where ordinary fermions live on the branes defined by the orbifold fixed-points and messenger fermions live in the bulk, are compared to those of ordinary four-dimensional flavour models, and some explicit examples are constructed.

In this chapter, we study the possibility of endowing the above-mentioned class of higher-dimensional models with a flavour symmetry of the Froggatt-Nielsen (FN) type [6]. This is done by introducing an extended flavour symmetry, which is then broken, as for the electroweak symmetry, by the combination of an orbifold projection and a SS twist. We focus on the model of ref. [22] and describe its minimal flavour extension. We show that by a wise choice of the flavour quantum numbers for bulk and brane fermion fields, it is possible to reproduce the observed pattern of the quark masses and CKM angles, although the mass obtained for the down quark tends to be too small, and observe that a similar approach is possible also for lepton masses and PMNS angles. The resulting model generates a 4D effective theory with a stabilized electroweak scale and a $U(1)$ FN symmetry.

The chapter is organized as follows. After quickly reviewing gauge-Higgs unification in sec. 7.1, we outline in sec. 7.2 the basic construction discussing explicitly a prototype model. In sec. 7.3 we generalize our construction to arbitrary representations of the electroweak and flavour groups. In sec. 7.4 we present more realistic examples of our construction. Finally, in sec. 7.5 we draw some conclusions and discuss future developments.

7.1 Gauge-Higgs unification in 5D

Our starting point is the model of gauge-Higgs unification described in ref. [22]. The basic physical idea is to break the electroweak symmetry in a non-local way, so that the Higgs mass is protected by the gauge invariance of the 5D theory. Indeed, in a 5D theory compactified on a circle with SS symmetry breaking, all ultraviolet (UV) divergent quantities at all orders in perturbation theory must be invariant under the full symmetries of the 5D theory [127]. This means that all symmetry-breaking quantities are finite, calculable and insensitive to the unknown UV dynamics. If one could find a 5D symmetry that forbid the Higgs mass term, a non-local breaking of this symmetry would protect the Higgs mass from any divergent radiative correction. Gauge-Higgs unification implements this idea, by identifying the Higgs boson with the internal component of a 5D gauge field, so that the 5D gauge symmetry protects the Higgs mass.

To construct a model of gauge-Higgs unification one must consider a gauge group large enough as to include 4D states corresponding to the $SU(2)_L \otimes U(1)_Y$ gauge bosons plus the Higgs doublet. The minimal possibility corresponds to an $SU(3)_W$ gauge group, broken first to $SU(2)_L \otimes U(1)_Y$ via a Z_2 orbifold projection, and then to $U(1)_{\rm em}$ with a SS twist.[1] The orbifold projection acting on the 5D gauge group leaves as 4D zero modes the SM gauge bosons plus a scalar doublet with the quantum numbers of the SM Higgs: the SS twist corresponds to a Vacuum Expectation Value (VEV) for the Higgs via the Hosotani mechanism. From the 4D point of view, this corresponds to the SM Higgs mechanism: however, higher-dimensional gauge invariance protects the Higgs mass. This remains true even though at the orbifold fixed points only the SM gauge group is present: indeed, the zero-modes of A_5 transform non-homogeneously under gauge transformations belonging to $SU(3)_W/(SU(2)_L \otimes U(1)_Y)$, so that the only possible counterterms are $SU(3)_W$-invariant ones [66] (see also [128]).

The price one has to pay for this UV insensitivity is the absence of a tree-level potential for the Higgs. This implies that the Higgs mass generated at one loop is generically too small (see ref. [22] for a detailed discussion of this problem). A related issue is the value of the SS twist that is dynamically generated: unless bulk fermions belonging to very high-rank representations of $SU(3)$ are present, one obtains twist parameters of order 10^{-1}, corresponding to an extra dimension of inverse radius $1/R \sim 10\, m_W \sim 1$ TeV, far

[1] In this way, one obtains $\sin^2\theta_W = 3/4$. An acceptable value of the weak mixing angle can be achieved by adding an extra $U(1)'$ factor and tuning its coupling relatively to the weak coupling, as done in ref. [22]. The additional $U(1)_X$ symmetry introduced in this way in the 4D effective theory is anomalous, and must therefore be spontaneously broken and decoupled.

below the LEP indirect bounds. Since these problems are unrelated to the issue of flavour symmetry breaking that will be discussed in this work, from now on we will assume that the value of the SS twist α is of order 10^{-2} thanks to some unspecified mechanism, so that $1/R \sim 10$ TeV and Kaluza-Klein (KK) excitations of electroweak gauge bosons do not pose any phenomenological problem.

As in the standard electroweak theory, the VEV of the Higgs field can induce a mass for the matter fermions. The relevant Yukawa couplings, however, originate in this case from the 5D gauge coupling. For bulk fermions, this implies that the Yukawa couplings are universal and their magnitude is simply the gauge coupling times a group-theoretical factor, depending only on the representation. Furthermore, no flavour symmetry breaking can arise from electroweak gauge couplings, so that one is left with a universal fermion mass and no flavour mixing. For brane fields localized at the orbifold fixed-points, on the contrary, the $SU(2)_L \times U(1)_Y$ symmetry would allow Yukawa couplings to be arbitrary and non-universal, but the non-linearly realized $SU(3)_W/(SU(2)_L \times U(1)_Y)$ symmetry implies that they all vanish. In order to achieve realistic Yukawa couplings, one is therefore led to consider the more general case of fermions that are a mixture of bulk and brane fields with wave functions depending non-trivially on the internal dimension [43]. This situation is most easily realized by considering bulk and brane fields that mix through non-universal bilinear couplings localized at the fixed-points [22]. The new eigenstates, resulting from the diagonalization of the quadratic Lagrangian for these fields, will then inherit non-vanishing and non-universal Yukawa couplings to the Higgs field. The structure of the mass couplings is pretty general, but their size is always at most of the order of the gauge coupling. This implies that the natural value of all the fermion masses induced in this way is of the order of m_W.

In the case where the above construction is realized with bulk fields that are much heavier than the brane fields, the lightest eigenstates are sharply localized fields whose dynamics is well approximated by an effective Lagrangian for the original brane fermions, obtained by integrating out the heavy bulk modes. From this perspective, the non-vanishing and non-universal Yukawa couplings for the light localized modes emerge as effective interactions induced through the exchange of the heavy bulk fermions, which have a non-vanishing but universal fundamental Yukawa coupling. This framework is very similar to the one occurring in models with flavour symmetries, the breaking of which is transmitted to the effective Yukawa couplings through a heavy fermion, and suggests that it should be possible to naturally generalize the model of ref. [22] to include a flavour symmetry.

The usual implementation of a FN $U(1)_F$ flavour symmetry goes as follows. One assigns a $U(1)_F$ charge to each of the SM fermions, and introduces some heavy vector-like fermions in order to construct gauge- and flavour-invariant Yukawa couplings. The flavour symmetry is then spontaneously broken by some VEV at a scale smaller than the mass of the heavy fermions, so that the effective Yukawa couplings for SM fermions generated at low energies are $Y_{IJ} \propto (\langle\phi\rangle/M)^{q_I - q_J}$, where $\langle\phi\rangle$ is the $U(1)_F$-breaking VEV, M is the mass of the heavy fermions, and q_I are the SM fermion flavour charges. Wave-function corrections and the potentially dangerous tree-level FCNC generated by the heavy fermions are power suppressed and negligible if the new particles live at a high scale.

It is then natural for us to consider the case of a $U(1)_F$ symmetry broken à la SS. Since rank lowering can only be achieved by combining an orbifold projection with a SS twist, we have to start from an $SU(2)_F$ symmetry in the bulk, broken to $U(1)_F$ by the orbifold and then to nothing via a SS twist. Clearly, since the mass scale of the heavy bulk fermions is around 10 TeV in the case of ref. [22], we should make sure that wave function corrections and tree-level FCNC couplings are under control. We have performed a preliminary analysis of this issue, which indicates that unwanted effects might indeed be kept sufficiently small with reasonable choices of parameters. A detailed analysis, together with a study of loop-induced FCNC's, is currently under way and will be presented elsewhere.

7.2 A prototype model

A minimal prototype of the models discussed in the previous section can be constructed as follows. The standard fermions are taken to live at the orbifold fixed-points, whereas the messenger fermions that activate the mechanism of symmetry breaking live in the bulk. A spontaneously broken Abelian flavour symmetry is then incorporated much in the same way as for the spontaneously broken electroweak symmetry, and both symmetry breakings are implemented at once by letting the orbifold projection and the SS twist act on both the electroweak and the flavour groups. The minimal choice of 5D flavour group allowing an Abelian group in the intermediate step and a full breaking in the final step is an $SU(2)_F$ group. For simplicity, we assume this to be a global symmetry, but the case of a local symmetry is similar. This flavour group is broken to a $U(1)_F$ subgroup through the orbifold projection, and finally to nothing through the SS twist.

The above construction is very general, and exploits for both the electroweak and the flavour symmetries the same minimal pattern of symmetry breaking discussed in ref. [129], which consists in first promoting the 4D group to a larger 5D group and then performing two non-commuting projections that enable to lower the rank. The standard fermions at the fixed-points form representations of $SU(2)_L \times U(1)_Y \times U(1)_F$, whereas the messenger fermions in the bulk form representations of $SU(3)_W \times SU(2)_F$. The construction can be applied in a perfectly similar way both to the quark and the lepton sectors. Here we shall focus on the quark sector. For the sake of clarity of presentation, we will first illustrate the general qualitative features of the construction with an explicit example, then generalize to arbitrary flavour charges and $SU(3)_W \otimes SU(2)_F$ representations, and finally discuss some realistic models.

7.2.1 Orbifold projection and SS twist

The projections defining the model are chosen as follows. The orbifold projection on a bulk field $\Phi_{\mathcal{R},\mathcal{R}'}$ in a generic representation $(\mathcal{R},\mathcal{R}')$ of $SU(3)_W \times SU(2)_F$ is taken to be

$$\Phi_{\mathcal{R},\mathcal{R}'}(x,-y) = \pm\big[P_L \otimes P_W^{\mathcal{R}} \otimes P_F^{\mathcal{R}'}\big]\, \Phi_{\mathcal{R},\mathcal{R}'}(x,y)\,, \tag{7.1}$$

where P_L depends on which Lorentz representation the field corresponds to ($P_L = 1$ for a scalar, $P_L = \gamma_5$ for a spinor, etc.) and P_W and P_F define the embedding of the projection into the weak and flavour groups. In order to achieve the desired symmetry breaking down to $SU(2)_W \times U(1)_Y \times U(1)_F$, we use the T_W^8 and T_F^3 generators[2] of $SU(3)_W$ and $SU(2)_F$ respectively, and choose:

$$P_W = e^{2i\pi\sqrt{3}T_W^8}\,, \qquad P_F = e^{-i\pi\,(d(T_F^3)-1)/2}\, e^{i\pi T_F^3}\,, \tag{7.2}$$

where $d(T)$ is the dimension of the matrix T acting on the representation $\mathcal{R}'$. The residual $SU(2)_F \times U(1)_Y$ electroweak gauge symmetries are associated with the generators T_W^a with $a = 1,2,3,8$ that commute with the projection: $[T_W^a, P_W] = 0$. Similarly, the surviving $U(1)_F$ flavour symmetry is associated to the only generator T_F^3 commuting with the projection: $[T_F^3, P_F] = 0$.

The Scherk-Schwarz twist on the generic representation $(\mathcal{R},\mathcal{R}')$ of $SU(3)_W \times SU(2)_F$ is similarly of the form:

$$\Phi_{\mathcal{R},\mathcal{R}'}(x, y + 2\pi R) = \Big[T_W^{\mathcal{R}}(\alpha) \otimes T_F^{\mathcal{R}'}(\beta)\Big]\; \Phi_{\mathcal{R},\mathcal{R}'}(x,y)\,, \tag{7.3}$$

where $T_W(\alpha)$ and $T_F(\beta)$ define the embedding of the twist into the weak and flavour groups and depend on two continuous parameters α and β. These must satisfy the usual consistency constraints $(T_W P_W)^2 = (T_F P_F)^2 = 1$ [45,46,94,95]. In order to further break by the twist the electroweak and flavour symmetries $SU(2)_F \times U(1)_Y \times U(1)_F$ preserved by the orbifold projection down to $U(1)_{\rm em}$, we use the T_W^6 and T_F^1 generators of $SU(3)_W$ and $SU(2)_F$ respectively, and choose:

$$T_W(\alpha) = e^{4\pi i\alpha T_W^6}\,, \qquad T_F(\beta) = e^{4\pi i\beta T_F^1}\,. \tag{7.4}$$

The residual $U(1)_{\rm em}$ electromagnetic gauge symmetry is associated with the only generator $T_W^3 + T_W^8/\sqrt{3}$ that commutes also with the twist: $[T_W^3 + T_W^8/\sqrt{3}, T_W] = 0$. Notice that

[2] We define the $SU(3)_W$ generators as $T^a = \lambda^a/2$, where λ^a are the standard Gell-Mann matrices with the normalization ${\rm Tr}\lambda^a\lambda^b = 2\delta^{ab}$. Similarly, we define the $SU(2)_F$ as $T^a = \sigma^a/2$, where σ^a are the standard Pauli matrices with the normalization ${\rm Tr}\sigma^a\sigma^b = 2\delta^{ab}$.

this fixes the hypercharge to be $Y = T_W^8/\sqrt{3}$. The flavour symmetry is instead completely broken since there is no generator commuting also with the twist.

The dimensionless quantities α and β are the order parameters for the rank-reducing breaking of the electroweak and flavour symmetries. Indeed, it is evident from eqs. (7.2) and (7.4) that the orbifold projection and the twist do not commute, that is $[P_W, T_W] \neq 0$ in the gauge sector and $[P_F, T_F] \neq 0$ in the flavour sector, unless $\alpha = n/2$ and $\beta = n/2$, with n integer. For the gauge symmetry, it is possible to relate the order parameter to the VEV of the Higgs field A_5 by performing a non-periodic gauge transformation that reabsorbs the twist [60–62]: $\alpha = g_5 R\langle A_5\rangle/2$. For the flavour symmetry, a similar relation would hold if it were local; the case where it is taken to be global can however be understood in a similar way by taking a suitable decoupling limit [127]. Notice finally that the electroweak and flavour symmetry breaking scales are naturally defined by $m_W = \alpha/R$ and $m_F = \beta/R$.

The effect of the SS twist on the orbifold-projected spectrum of KK modes of bulk fields will as usual amount to shifting the standard integer-moded masses $m_n = n/R$ obtained for fields that are periodic along the internal circle S^1 with radius R through a quantity that depends on the symmetry breaking parameters α and β. To be more precise, notice that the generators appearing in the exponents of the orbifold projection and SS twist do not commute. Starting from the standard basis in which the Cartan generators T_W^8 and T_F^3 appearing in the orbifold projection are diagonal, the generators T_W^6 and T_F^1 appearing in the twist can be brought into diagonal forms, which we denote by t_W and t_F, through some suitable unitary transformations U_W and V_F:

$$t_W = U_W T_W^6 U_W^\dagger\,, \qquad t_F = V_F T_F^1 V_F^\dagger\,. \tag{7.5}$$

In the transformed basis where the SS twist is diagonal (but the orbifold projection is not diagonal), the mass spectrum can be written in terms of the entries of the diagonalized twist generator simply as $m_n(\alpha,\beta) = (n + 2t_W\alpha + 2t_F\beta)/R$ (see sec. 7.2.5).

7.2.2 Field content

The field content of the model is a generalization of the one considered in ref. [22], where now all the brane fields must not only belong to $SU(2)_L \times U(1)_Y$ representations but also have definite charges under the $U(1)_F$ subgroup, and similarly all the bulk fields must also belong not only to $SU(3)_W$ but also to $SU(2)_F$ representations. Notice that the charge under the $U(1)_F$ flavour group preserved by the orbifold projection is quantized, as a consequence of the fact that the original flavour group is non-Abelian, and represented by $q_F = T_F^3$. This constrains in an interesting way the allowed charge assignments for the brane fields. The minimal content of brane and bulk fields that is required in order to construct the flavour extension of the model of ref. [22] is then quite rigidly fixed.

The SM fermions are introduced as brane fields at the fixed-points of the orbifold projection. Denoting by y the periodic coordinate of the extra dimension, the two fixed-points are located at $y = 0$ and $y = \pi R$ and represent the two boundaries of the physical space, the segment $[0, \pi R]$ in the extra dimension. Each of the left- and right-handed fields can be located at any of the two fixed-points. The precise distribution that is chosen is qualitatively not too important as far as the low-energy effective theory is concerned, but it is quite relevant for the consistency of the theory, and in particular for the issue of anomalies. Indeed, it is known that globally vanishing localized anomalies occur in theories with a generic content of bulk and brane fields and that requiring their cancellation may have non-trivial implications on the theory [130–133] (see [134] for a general review). The issue of localized anomalies has already been discussed in ref. [22], and since the flavor extension examined here does not involve any novelty in this respect, we shall not discuss it any further here. For simplicity, we assume that all the left-handed fields are located at $y = 0$ and the right-handed ones at $y = \pi R$, and their interactions are constrained to be invariant under the residual symmetries described above. We introduce the following representations of $SU(2)_L \times U(1)_Y \times U(1)_F$:

- Left-handed fields localized at $y = 0$:

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} : \mathbf{2}_{\frac{1}{6},q} \quad \text{or equivalently} \quad Q_R^c = \begin{pmatrix} d_R^c \\ \text{-}u_R^c \end{pmatrix} = \mathbf{2}_{-\frac{1}{6},-q} \,. \tag{7.6}$$

- Right-handed fields localized at $y = \pi R$:

$$\begin{aligned} u_R &= \mathbf{1}_{\frac{2}{3},u} \,, \quad \text{or equivalently} \;\; \text{-}u_L^c = \mathbf{1}_{-\frac{2}{3},-u} \,, \\ d_R &= \mathbf{1}_{-\frac{1}{3},d} \,, \quad \text{or equivalently} \;\; d_L^c = \mathbf{1}_{\frac{1}{3},-d} \,, \end{aligned} \tag{7.7}$$

with the notation $\mathbf{R}_{q_Y,q_F}$, where $\mathbf{R}$ is the $SU(2)_L$ representation and q_Y and q_F are the $U(1)_Y$ and $U(1)_F$ charges respectively. As a first example, we choose the charge assignment reported in Table 7.1.

Field	q_F	Field	q_F	Field	q_F
Q_{1L}	4	d_{1R}	-1	u_{1R}	-4
Q_{2L}	3	d_{2R}	0	u_{2R}	-1
Q_{3L}	1	d_{3R}	1	u_{3R}	1

Table 7.1: *Flavour charges of SM fermions.*

The bulk fields consist of the 5D gauge fields and of the heavy fermions that are needed to induce the effective Yukawa couplings as in ref. [22]. The rôle played by the gauge fields has been extensively explained in ref. [22] and will not be discussed again here. The only novelty concerns the heavy messenger fermions, which will now carry flavour quantum numbers. We introduce two pairs $l = u, d$ of fermion fields $(\psi^l, \tilde{\psi}^l)$ with opposite orbifold parities, with a bulk mass term that makes all their modes heavy. Following ref. [22], we take these two pairs to be weak triplets to generate masses for down-type quarks, and weak sixplets to generate masses for up-type quarks. Concerning the representation under $SU(2)_F$, from Table 7.1 we see that Q_L and u_L^c have flavour charges with absolute value up to four: the minimal choice is therefore a nineplet, which contains fields with $U(1)_F$ charges from -4 to 4. Summarizing, we have bulk fields in the following representations of $SU(3)_W \times SU(2)_F$:

- Bulk fields with negative overall parity:

$$\psi^d : \Big(\mathbf{3}, \mathbf{9}\Big) , \quad \tilde{\psi}^u : \Big(\mathbf{6}, \mathbf{9}\Big) , \tag{7.8}$$

- Bulk fields with positive overall parity:

$$\tilde{\psi}^d : \Big(\mathbf{3}, \mathbf{9}\Big) , \quad \psi^u : \Big(\mathbf{6}, \mathbf{9}\Big) . \tag{7.9}$$

The decomposition of the above representations of the $SU(3)_W \times SU(2)_F$ group under its $SU(2)_Y \times U(1)_Y \times U(1)_F$ subgroup, which we will need to determine the coupling of the bulk fields to the brane fields, has the following form:

$$\begin{aligned} \Big(\mathbf{3}, \mathbf{9}\Big) &\rightarrow \mathbf{2}_{\frac{1}{6},q} \oplus \mathbf{1}_{-\frac{1}{3},d} , \\ \Big(\mathbf{6}, \mathbf{9}\Big) &\rightarrow \mathbf{3}_{\frac{1}{3},Q} \oplus \mathbf{2}_{-\frac{1}{6},-q} \oplus \mathbf{1}_{-\frac{2}{3},-u} , \end{aligned} \tag{7.10}$$

with Q, q, u, d ranging from -4 to 4. The only components that have the right quantum numbers to couple to the brane fermions are the $SU(2)_W$ doublets and singlets, with $U(1)_F$ charges matching the SM ones given in Table 7.1.

The action of the orbifold projection on the bulk fermion fields is given by

$$\begin{aligned} P_W^{\mathbf{3}} &= \mathrm{diag}(-1, -1, 1) , \quad P_W^{\mathbf{6}} = \mathrm{diag}(1, 1, -1, 1, -1, 1) , \\ P_F^{\mathbf{9}} &= \mathrm{diag}(1, -1, 1, -1, 1, -1, 1, -1, 1) . \end{aligned} \tag{7.11}$$

This implies that after the projection the particle content is given by an electroweak doublet and an electroweak singlet with flavour charges ranging from -4 to 4, belonging to ψ^l if the flavour charge is even and to $\tilde{\psi}^l$ if it is odd. In other words, one and only one of the two bulk fields ψ^l and $\tilde{\psi}^l$ always has a component with the right quantum numbers to couple to the SM brane fermions.

The choice of the $SU(3)_W$ representation for the messenger fermions in the bulk influences only the overall magnitude of the induced Yukawa couplings, whereas the choice of the $SU(2)_F$ representation for these bulk fermions, together with the $U(1)_F$ charges for the matter brane fermions, determines the flavour structure.

7.2.3 Lagrangian

The structure of the Lagrangian is the same as in ref. [22]: in addition to the kinetic terms for the bulk and brane fields, we introduce an arbitrary bilinear mixing between them. The couplings of the three generations of left- and right-handed brane fields Q_L, u_R, d_R and their conjugates to the bulk fields ψ^l or $\tilde{\psi}^l$ are parametrized by couplings e_L^l and e_R^l with mass-dimension 1/2, in each sector $l = u, d$. Each brane field can couple either to ψ^l or $\tilde{\psi}^l$, and has therefore only one relevant coupling. To write these couplings more explicitly, it is convenient to embed the brane fields into new fields $\chi_{L,R}^{u,d}$, $\tilde{\chi}_{L,R}^{u,d}$ which have the same matrix structure as the representations of $SU(3)_W \times SU(2)_F$ to which the bulk fields belong, the extra entries being filled with zeroes, and then further combine left and right components into Dirac fields: $\chi^{u,d} = \chi_L^{u,d} + \chi_R^{u,d}$ and $\tilde{\chi}^{u,d} = \tilde{\chi}_L^{u,d} + \tilde{\chi}_R^{u,d}$. Correspondingly, it is convenient to embed the diagonal matrices of couplings e_1^l and e_2^l in family space into new diagonal matrices of couplings $\hat{e}_1^l$ and $\hat{e}_2^l$ in flavour space. In our example, we have:

$$\chi^d = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_W \otimes \begin{pmatrix} u_L^1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_F + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_W \otimes \begin{pmatrix} d_L^1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_F + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_W \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ d_R^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_F , \quad (7.12)$$

$$\tilde{\chi}^d = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_W \otimes \begin{pmatrix} 0 \\ u_L^2 \\ 0 \\ u_L^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_F + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_W \otimes \begin{pmatrix} 0 \\ d_L^2 \\ 0 \\ d_L^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_F + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_W \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ d_R^3 \\ 0 \\ d_R^1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_F , \quad (7.13)$$

$$\chi^u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}_W \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \text{-}u_R^{c1} \end{pmatrix}_F + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_W \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ d_R^{c1} \end{pmatrix}_F + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_W \otimes \begin{pmatrix} u_L^{c1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_F , \quad (7.14)$$

$$\tilde{\chi}^u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}_W \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \text{-}u_R^{c3} \\ 0 \\ \text{-}u_R^{c2} \\ 0 \end{pmatrix}_F + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_W \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ d_R^{c3} \\ 0 \\ d_R^{c2} \\ 0 \end{pmatrix}_F + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_W \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_L^{c2} \\ 0 \\ u_L^{c3} \\ 0 \\ 0 \\ 0 \end{pmatrix}_F , \quad (7.15)$$

with $u^{1,2,3}$ and $d^{1,2,3}$ denoting the three generation quarks, and

$$\begin{aligned}
\hat{e}_1^d &= \mathrm{diag}(e_{1,1}^d, e_{1,2}^d, 0, e_{1,3}^d, 0, 0, 0, 0, 0)\,, \\
\hat{e}_2^d &= \mathrm{diag}(0, 0, 0, e_{2,3}^d, e_{2,2}^d, e_{2,1}^d, 0, 0, 0)\,, \\
\hat{e}_1^u &= \mathrm{diag}(0, 0, 0, 0, 0, e_{1,3}^u, 0, e_{1,2}^u, e_{1,1}^u)\,, \\
\hat{e}_2^u &= \mathrm{diag}(e_{2,1}^u, 0, 0, e_{2,2}^u, 0, e_{2,3}^u, 0, 0, 0)\,.
\end{aligned} \tag{7.16}$$

With this notation, and discarding irrelevant operators, which give negligible physical effects at low energies, the most general local Lagrangian for the light SM fields and the heavy flavour messengers that is compatible with the symmetries of the theory has the structure

$$\mathcal{L} = \mathcal{L}^{\mathrm{bulk}} + \delta(y)\mathcal{L}^0 + \delta(y-\pi R)\mathcal{L}^{\pi R}\,, \tag{7.17}$$

with

$$\mathcal{L}^{\mathrm{bulk}} = \sum_{l=u,d}\Big[i\bar{\psi}^l\gamma^M D_M\psi^l + i\bar{\tilde{\psi}}^l\gamma^M D_M\tilde{\psi}^l - M_l(\bar{\psi}^l\tilde{\psi}^l + \bar{\tilde{\psi}}^l\psi^l)\Big]\,, \tag{7.18}$$

$$\begin{aligned}
\mathcal{L}^0 &= i\bar{\chi}_L^d\gamma^\mu D_\mu\chi_L^d + i\bar{\tilde{\chi}}_L^d\gamma^\mu D_\mu\tilde{\chi}_L^d + i\bar{\chi}_R^u\gamma^\mu D_\mu\chi_R^u + i\bar{\tilde{\chi}}_R^u\gamma^\mu D_\mu\tilde{\chi}_R^u \\
&\quad + \Big[\bar{\chi}_L^d\,\hat{e}_1^{d\dagger}\,\psi^d + \bar{\tilde{\chi}}_L^d\,\hat{e}_1^{d\dagger}\,\tilde{\psi}^d + \bar{\chi}_R^u\,\hat{e}_1^{u\dagger}\,\psi^u + \bar{\tilde{\chi}}_R^u\,\hat{e}_1^{u\dagger}\,\tilde{\psi}^u + \mathrm{h.c.}\Big]\,,
\end{aligned} \tag{7.19}$$

$$\begin{aligned}
\mathcal{L}^{\pi R} &= i\bar{\chi}_L^u\gamma^\mu D_\mu\chi_L^u + i\bar{\tilde{\chi}}_L^u\gamma^\mu D_\mu\tilde{\chi}_L^u + i\bar{\tilde{\chi}}_R^d\gamma^\mu D_\mu\tilde{\chi}_R^d + i\bar{\chi}_R^d\gamma^\mu D_\mu\chi_R^d \\
&\quad + \Big[\bar{\chi}_R^d\,\hat{e}_2^{d\dagger}\,\psi^d + \bar{\tilde{\chi}}_R^d\,\hat{e}_2^{d\dagger}\,\tilde{\psi}^d + \bar{\chi}_L^u\,\hat{e}_2^{u\dagger}\,\psi^u + \bar{\tilde{\chi}}_L^u\,\hat{e}_2^{u\dagger}\,\tilde{\psi}^u + \mathrm{h.c.}\Big]\,.
\end{aligned} \tag{7.20}$$

We have here tacitly excluded the possibility that odd operators might appear in the Lagrangian with coefficients that are themselves odd functions of the coordinates, behaving as constants in the bulk and jumping discontinuously at the branes. This is reasonable, since this kind of odd operators can be distinguished from ordinary even operators by a local parity symmetry [133]. It should be noticed, however, that imposing the latter symmetry significantly restricts the possibilities for canceling potential localized anomalies, since it forbids bulk Chern-Simons counterterms.

7.2.4 Structure of the induced couplings

As in ref. [22], effective Yukawa couplings, wave-function and vertex corrections for the standard matter fermions are generated in the low-energy effective theory defined by integrating out the heavy messenger fermions. For example, mass terms are obtained from the diagrams in Fig. 1. In this case, however, a given brane fermion can couple only to the flavour component of the bulk fermions that has the same $U(1)_F$ charge. This implies that a non-vanishing Yukawa coupling, wave-function or vertex correction

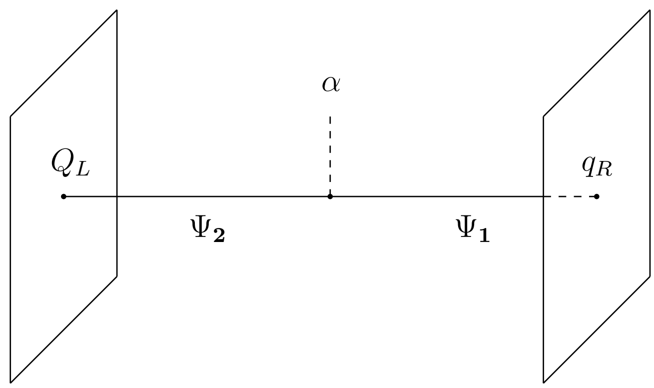

Figure 7.1: *Diagram inducing the effective mass in the presence of $SU(3)_W$ gauge symmetry breaking only: all the fields carry the same flavour charge. The insertion of α switches from the doublet to the singlet components of the bulk field. Here Q_L and q_R can be any left- and right-handed brane fermion and Ψ represents the pair of bulk fermions.*

is generated only if the involved brane fields have equal $U(1)_F$ charge, as long as the $U(1)_F$ symmetry stays unbroken, that is for $\beta = 0$. The other Yukawa couplings, wave-function and vertex corrections, involving brane fields with different $U(1)_F$ charges, can be generated only if the $U(1)_F$ symmetry is broken, that is $\beta \neq 0$. In this case, we have the diagrams in Fig. 2. Since $T_F^1 = (T_F^+ + T_F^-)/2$ can change the $U(1)_F$ charge by 1 unit, in order to connect two brane fields with charges differing by some integer k, we need $|k|$ insertions of βT_F^1. The effect will thus be of order $\beta^{|k|}$.

Actually, a further restriction turns out to be present, depending on whether k is even or odd, as a consequence of the fact that the two types of bulk fermions ψ^l and $\tilde{\psi}^l$ can couple only to SM fermions with even and odd flavour charges respectively (in the example under consideration). The Yukawa couplings can be generated through the exchange of bulk fermions with an even or odd number of bulk mass insertions, *i.e.* with or without a $\psi^l \Leftrightarrow \tilde{\psi}^l$ transition. Non-vanishing entries can therefore be generated only with even or odd k, depending on whether the involved fields couple to the same or to a different kind of bulk fields ψ or $\tilde{\psi}$. Wave-function and vertex corrections can instead be generated only with an even number of bulk mass insertions, *i.e.* without an overall $\psi^l \Leftrightarrow \tilde{\psi}^l$ transition, and a non-vanishing correction is therefore generated only for k even.

The above reasoning shows that with a suitable assignment of the $SU(2)_F$ quantum numbers for brane and bulk fermions, it is possible to induce effective mass matrices with a pattern of matrix elements that can naturally explain the hierarchies among observed masses and mixing angles for matter fermions. Just as with the FN mechanism, the entries of the Yukawa couplings $Y_{IJ}^{u,d}$ (from now on we denote by I and J the family

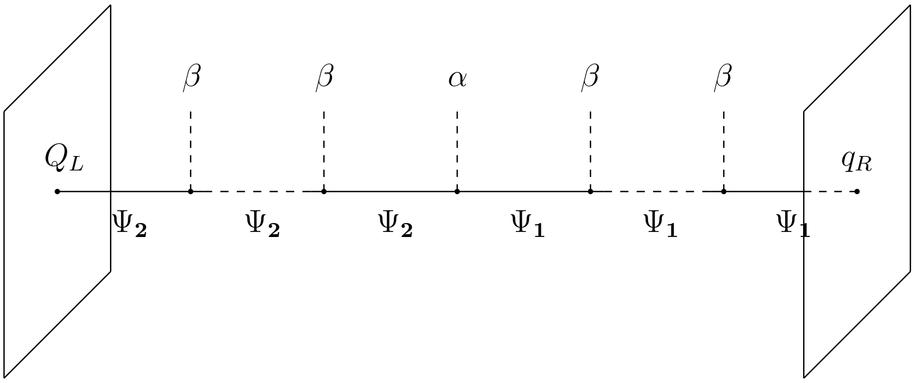

Figure 7.2: *Diagram inducing the effective mass in the presence of both gauge and flavour symmetry breaking. Each insertion of β switches between two components of bulk fields with flavour charges differing by one unit. The minimal number of such insertions that is needed to get a non-vanishing result is equal to the difference between the flavour charges of the left- and right-handed brane fields. Moreover, if this number is even, there is no mass insertion for the bulk fields, whereas when it is odd, there must be one mass insertion. Here Q_L and q_R can be any left- and right-handed brane fermion and Ψ represents the pair of bulk fermions.*

index) and the wave-function factors Z^Q_{IJ} and $Z^{u,d}_{IJ}$ for doublets and singlets respectively, can be expressed as powers of the order parameter $\lambda \equiv \pi\beta$ for the breaking of the Abelian flavour symmetry, modulo numerical factors of order one. The results can be written in terms of the charges q_I of the left-handed doublets Q and the charges l_I of the right-handed singlets $l = u, d$ as:

$$Y^l_{IJ} \sim \lambda^{|q_I - l_J|}\,, \tag{7.21}$$

$$Z^Q_{IJ} \sim \begin{cases} \delta_{IJ} + \lambda^{|q_I - q_J|} & \text{for } |q_I - q_J| \text{ even} \\ \delta_{IJ} \quad \text{for } |q_I - q_J| \text{ odd} \end{cases}\,, \tag{7.22}$$

$$Z^l_{IJ} \sim \begin{cases} \delta_{IJ} + \lambda^{|l_I - l_J|} & \text{for } |l_I - l_J| \text{ even} \\ \delta_{IJ} \quad \text{for } |l_I - l_J| \text{ odd} \end{cases}\,. \tag{7.23}$$

The physical Yukawa couplings are obtained after performing a transformation on matter fermions that brings their kinetic terms to a canonical form. To do so (see for example [1, 8, 135–138]), we first diagonalize the wave functions as $Z^Q = U^{Q\dagger} D^Q U^Q$ and $Z^l = U^{l\dagger} D^l U^l$ in terms of some unitary matrices U^Q and U^l. In general, the diagonal matrices have entries of order one, $D^Q_{II} \sim 1$ and $D^l_{II} \sim 1$, but differ from the identity, while the U matrices have the same form as the wave-function corrections themselves, *i.e.*

$U^Q_{IJ} \sim \delta_{IJ} + \lambda^{|q_I - q_J|}$ and $U^l_{IJ} \sim \delta_{IJ} + \lambda^{|l_I - l_J|}$. We then redefine the matter fields to be $\hat{Q} = \sqrt{D^Q} U^Q Q$ and $\hat{l} = \sqrt{D^l} U^l l$. In this way, the new wave-function factors are $\hat{Z}^q = 1$ and $\hat{Z}^l = 1$, whereas the new Yukawa coupling is given by $\hat{Y}^l = (D^Q)^{-\frac{1}{2}} U^Q Y^l U^{l\dagger} (D^l)^{-\frac{1}{2}}$. In terms of λ this means

$$\hat{Y}^l_{IJ} \sim \sum_{KL} \lambda^{|q_I - q_K| + |q_K - l_L| + |l_L - l_J|} \sim \lambda^{|q_I - l_J|} \, . \tag{7.24}$$

The last step, which follows from the inequality $|x| + |y| \geq |x + y|$, shows that as in standard 4D flavour models the wave function corrections do not mess up the structure of the Yukawa couplings.

Equations (7.24) realize the starting point for building interesting flavour models. However, a more careful analysis shows that our 5D construction presents a number of peculiarities that make it much more constrained than a generic 4D flavour model of the FN type, mostly due to the embedding in a non-Abelian group and to the structure of the mediator sector:

- The flavour charge is quantized and charge differences are integer.
- The precise numerical factors appearing in the induced Yukawa couplings are correlated, and contain potentially large group-theoretical coefficients.

7.2.5 Effective Lagrangian and induced couplings

We now present the explicit computation of the 4D effective Lagrangian, and in particular the corrections to the kinetic and mass terms for the SM fields. The leading effects are obtained by integrating out the heavy bulk fermions at the classical level. The computation can be done along the lines of ref. [22]. In order to illustrate the procedure, we start by discussing in detail the d-quark sector. The up quark sector will then be easily explained.

Mode decomposition

In general, matter fields obey the compactification conditions in eqs. (7.1) and (7.3). In the following, $SU(3)_W$ and $SU(2)_F$ indices will be denoted by $i, j, \ldots$ and $a, b, \ldots$ respectively, and we work in a basis where the orbifold projection is diagonal, whereas, in general, the twist is non diagonal. For fixed electroweak and flavour indices, the γ_5 matrix acting in the orbifold projection causes the right- and left-handed matter field components to have different parity. Hence we can write the matter field components as follows:

$$\psi_{i,a}(x,y) = \psi^+_{i,a}(x,y) + \psi^-_{i,a}(x,y) \, , \tag{7.25}$$

where $\psi^+_{i,a}(x,y)$ and $\psi^-_{i,a}(x,y)$ are fields with positive and negative orbifold parity respectively and a given chirality which depends on i and a. Thus the superscript $\pm$ refers to the orbifold parity. The fields satisfying the condition (7.1) can be expanded in KK modes as

$$\begin{aligned} \psi^+_{i,a}(x,y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{+\infty} (\frac{1}{\sqrt{2}})^{\delta_{n,0}} (\psi^+_{i,a})_n(x) \ \cos(\frac{ny}{R}) \ , \\ \psi^-_{i,a}(x,y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{+\infty} (\psi^-_{i,a})_n(x) \ \sin(\frac{ny}{R}) \ . \end{aligned} \tag{7.26}$$

It is convenient to express $\psi^+_{i,a}(x,y)$, $\psi^-_{i,a}(x,y)$ as sums over all integer modes, both positive and negative; this is done by defining the negative modes of a given component as reflection of the positive modes: $(\psi^\pm_{i,a})^\dagger_{-n}(x) = \pm(\psi^\pm_{i,a})_n(x)$. The new mode expansion for untwisted fields is then

$$\begin{aligned} \psi^+_{i,a}(x,y) &= \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{+\infty} \eta_n (\psi^+_{i,a})_n(x) \cos(\frac{ny}{R}) \ , \\ \psi^-_{i,a}(x,y) &= \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{+\infty} \eta_n (\psi^-_{i,a})_n(x) \sin(\frac{ny}{R}) \ , \end{aligned} \tag{7.27}$$

where

$$\eta_n = \begin{cases} 1/\sqrt{2} & \text{if} \quad n \neq 0 \\ 1 & \text{if} \quad n = 0 \end{cases} \ . \tag{7.28}$$

We now switch to the basis in which the SS twist is diagonal. The eigenvectors $\Psi^\pm_{i,a}$ of the twist $T_{\mathcal{R},\mathcal{R}'}$ can be written as follows:

$$\Psi^+_{i,a} = (U_W)^{\mathcal{R}}_{ij} (V_F)^{\mathcal{R}'}_{ab} \psi^+_{j,b} \ , \qquad \Psi^-_{i,a} = (U_W)^{\mathcal{R}}_{ij} (V_F)^{\mathcal{R}'}_{ab} \psi^-_{j,b} \ , \tag{7.29}$$

where $(U_W)^{\mathcal{R}}$ and $(V_F)^{\mathcal{R}'}$ are two unitary matrices in the gauge and flavour space and the labels $\mathcal{R}$ and $\mathcal{R}'$ denote the representation to which the matter fields belong. Since the rotation mixes different indices i and a, corresponding to different chiralities, the fields $\Psi^+_{i,a}$ and $\Psi^-_{i,a}$ do not have a definite chirality, when expanded in KK modes as in eqs. (7.27). On the other hand, since the twist mixes fields with the same orbifold parity, it is possible to diagonalize it with transformations acting separately on ψ^+ and ψ^-.

In the new basis, the twist is diagonal. The explicit expressions for the unitary matrices $(U_W)^{\mathbf{3}}$, $(U_W)^{\mathbf{6}}$ and $(V_F)^{\mathbf{9}}$ that diagonalize the twist matrices $(T^6_W)^{\mathbf{3}}$, $(T^6_W)^{\mathbf{6}}$ and $(T^1_W)^{\mathbf{9}}$ to the forms $(t_W)^{\mathbf{3}} = \mathrm{diag}(1/2, 0, -1/2)$, $(t_W)^{\mathbf{6}} = \mathrm{diag}(1, 1/2, 0, 0, 0, -1/2, -1)$ and $(t_F)^{\mathbf{9}} =$

diag$(-4,4,-3,3,-2,2,-1,1,0)$ are given by:

$$(U_W)^{\mathbf{3}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \\ \sqrt{2} & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad (U_W)^{\mathbf{6}} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 & \sqrt{2} & 1 \\ 0 & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 & \sqrt{2} \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\sqrt{2} & 1 \end{pmatrix},$$

$$(V_F)^{\mathbf{9}} = \frac{1}{16} \begin{pmatrix} 1 & -\sqrt{8} & \sqrt{28} & -\sqrt{56} & \sqrt{70} & -\sqrt{56} & \sqrt{28} & -\sqrt{8} & 1 \\ 1 & \sqrt{8} & \sqrt{28} & \sqrt{56} & \sqrt{70} & \sqrt{56} & \sqrt{28} & \sqrt{8} & 1 \\ -\sqrt{8} & 6 & -\sqrt{56} & \sqrt{28} & 0 & -\sqrt{28} & \sqrt{56} & -6 & \sqrt{8} \\ -\sqrt{8} & -6 & -\sqrt{56} & -\sqrt{28} & 0 & \sqrt{28} & \sqrt{56} & 6 & \sqrt{8} \\ \sqrt{28} & -\sqrt{56} & 4 & \sqrt{8} & -\sqrt{40} & \sqrt{8} & 4 & -\sqrt{56} & \sqrt{28} \\ \sqrt{28} & \sqrt{56} & 4 & -\sqrt{8} & -\sqrt{40} & -\sqrt{8} & 4 & \sqrt{56} & \sqrt{28} \\ -\sqrt{56} & \sqrt{28} & \sqrt{8} & -6 & 0 & 6 & -\sqrt{8} & -\sqrt{28} & \sqrt{56} \\ -\sqrt{56} & -\sqrt{28} & \sqrt{8} & 6 & 0 & -6 & -\sqrt{8} & \sqrt{28} & \sqrt{56} \\ \sqrt{70} & 0 & -\sqrt{40} & 0 & 6 & 0 & -\sqrt{40} & 0 & \sqrt{70} \end{pmatrix}. \quad (7.30)$$

Let us now define

$$\hat{\Psi}_{i,a} = \left(\Psi^+_{i,a}, \Psi^-_{i,a}\right) \ . \quad (7.31)$$

In this basis, the effect of the twist amounts to shifting the masses of the KK modes by the quantity $2(t_W)_{ii}\alpha + 2(t_F)_{aa}\beta$. Therefore, suppressing all the indices, the new KK mass spectrum is given by

$$m_n(\alpha,\beta) = \frac{n\sigma_1 + (2t_W\alpha + 2t_F\beta)\mathbb{1}}{R}\,, \quad (7.32)$$

where σ_1 and $\mathbb{1}$ act at fixed i,a on the space $(\Psi^+_{i,a}, \Psi^-_{i,a})$ and connect terms with opposite and equal orbifold parity respectively. Finally, a complete diagonalization can be achieved by mixing states with opposite orbifold chirality:

$$(\Psi_{i,a})_n = \eta_n \left[(\Psi^+_{i,a})_n + (\Psi^-_{i,a})_n\right] \quad (7.33)$$

where now positive and negative n components of $\Psi_{i,a}$ are independent, and their mass is simply given by

$$m_n(\alpha,\beta) = \frac{n + (2t_W\alpha + 2t_F\beta)}{R}\,. \quad (7.34)$$

Construction of the effective Lagrangian

In order to derive the effective Lagrangian that is induced for the SM fermions by integrating out the bulk fermions at the classical level, we use for the latter the mode decomposition derived in previous subsection, and switch to 4D momentum space. The relevant linear and quadratic parts of the Lagrangian for the modes of the bulk fermions then becomes

$$\mathcal{L} = \sum_{n=-\infty}^{\infty} \Big[\mathcal{L}_n^{\rm bulk} + \mathcal{L}_n^0 + (-1)^n \mathcal{L}_n^{\pi R}\Big]\,, \tag{7.35}$$

where

$$\mathcal{L}_n^{\rm bulk} = \sum_{l=u,d} \Big[\bar\Psi_n^l\big(\not p - m_n\big)\Psi_n^l + \bar{\tilde\Psi}_n^l\big(\not p + m_n\big)\tilde\Psi_n^l - M_l\Big(\bar\Psi_n^l\tilde\Psi_n^l + \bar{\tilde\Psi}_n^l\Psi_n^l\Big)\Big]\,, \tag{7.36}$$

$$\begin{aligned}\mathcal{L}_n^0 = \frac{1}{\sqrt{2\pi R}}\Big[&\bar\chi_L^d(U_W V_F\,\hat e_1^d)^\dagger\Psi_n^d + \bar{\tilde\chi}_L^d(U_W V_F\,\hat e_1^d)^\dagger\tilde\Psi_n^d \\ &+ \bar\chi_R^u(U_W V_F\,\hat e_1^u)^\dagger\Psi_n^u + \bar{\tilde\chi}_R^u(U_W V_F\,\hat e_1^u)^\dagger\tilde\Psi_n^u + \text{h.c.}\Big]\,, \end{aligned}\tag{7.37}$$

$$\begin{aligned}\mathcal{L}_n^{\pi R} = \frac{1}{\sqrt{2\pi R}}\Big[&\bar\chi_R^d(U_W V_F\,\hat e_2^d)^\dagger\Psi_n^d + \bar{\tilde\chi}_R^d(U_W V_F\,\hat e_2^d)^\dagger\tilde\Psi_n^d \\ &+ \bar\chi_L^u(U_W V_F\,\hat e_2^u)^\dagger\Psi_n^u + \bar{\tilde\chi}_L^u(U_W V_F\,\hat e_2^u)^\dagger\tilde\Psi_n^u + \text{h.c.}\Big]\,. \end{aligned}\tag{7.38}$$

From these expressions it is clear that the physics of the light modes depends on the mass mixings $\hat e_{1,2}/\sqrt{2\pi R}$ encoding the couplings between brane and bulk modes and on the masses M_l for the bulk modes. The relevant dimensionless parameters are then the products of these masses with the length πR of the internal dimension:

$$\epsilon_{1,2}^l = \sqrt{\pi R/2}\,\hat e_{1,2}^l\,, \qquad x_l = \pi R M_l\,. \tag{7.39}$$

For convenience, we also define the ϵ couplings in the basis of diagonal twist:

$$\varepsilon_{1,2}^l = U_W V_F \epsilon_{1,2}^l\,. \tag{7.40}$$

The mass and wave-function corrections are generated by diagrams similar to the ones in Figs. 1 and 2. The result depends on the vacuum expectation value of the Higgs field A_5 through the dimensionless parameter $\alpha = g_5 R\langle A_5\rangle/2$, and on the phase β induced by the twist. If the flavour symmetry were local, β would be related to the fifth component of the corresponding $SU(2)_F$ field. For a global symmetry, β is a free parameter related to the phase accumulated by the twist. The tree-level propagator in momentum space for the KK modes of Ψ^l and $\tilde\Psi^l$ is given, in two-by-two matrix notation, by the following expression:

$$S_l^n = \frac{i}{p^2 - m_n(\alpha,\beta)^2 - (M_l)^2}\begin{pmatrix} \not p + m_n(\alpha,\beta) & M_l \\ M_l & \not p - m_n(\alpha,\beta)\end{pmatrix}\,. \tag{7.41}$$

The effective action is then obtained by integrating out the bulk fermions at the classical level, using the above propagator and treating the brane fields as sources, as in ref. [22]. We find a contribution to the effective Lagrangian in momentum space of the form $\mathcal{L}_d^{\rm eff} = \mathcal{L}_d^{\rm kin} + \mathcal{L}_d^{\rm m}$, where $\mathcal{L}_d^{\rm kin}$ contains the kinetic term corrections of SM matter fields and is given by

$$
\begin{aligned}
\mathcal{L}_d^{\rm kin} = \; & \overline{\chi}_L^d \varepsilon_1^{d^\dagger} \; (\not{p}) \; F(p, M_d, 2t_W\alpha + 2t_F\beta) \; \varepsilon_1^d \chi_L \\
& + \overline{\chi}_R^d \varepsilon_2^{d^\dagger} \; (\not{p}) \; F(p, M_d, 2t_W\alpha + 2t_F\beta) \; \varepsilon_2^d \chi_R^d \\
& + \overline{\tilde{\chi}}_L^d \varepsilon_1^{d^\dagger} \; (\not{p}) \; F(p, M_d, 2t_W\alpha + 2t_F\beta) \; \varepsilon_1^d \tilde{\chi}_L^d \\
& + \overline{\tilde{\chi}}_R^d \varepsilon_2^{d^\dagger} \; (\not{p}) \; F(p, M_d, 2t_W\alpha + 2t_F\beta) \; \varepsilon_2^d \tilde{\chi}_R^d \; ,
\end{aligned}
\tag{7.42}
$$

whereas $\mathcal{L}_d^{\rm m}$ contains the effective mass terms and is given by

$$
\begin{aligned}
\mathcal{L}_d^{\rm m} = \frac{1}{\pi R} \Big\{ & \Big[\overline{\chi}_L^d \varepsilon_1^{d^\dagger} \; G_1(p, M_d, 2t_W\alpha + 2t_F\beta) \; \varepsilon_2^d \chi_R^d + \text{h.c.} \Big] \\
& - \Big[\overline{\tilde{\chi}}_L^d \varepsilon_1^{d^\dagger} \; G_1(p, M_d, 2t_W\alpha + 2t_F\beta) \; \varepsilon_2^d \tilde{\chi}_R^d + \text{h.c.} \Big] \\
& + \Big[\overline{\chi}_L^d \varepsilon_1^{d^\dagger} \; G_2(p, M_d, 2t_W\alpha + 2t_F\beta) \; \varepsilon_2^d \tilde{\chi}_R^d + \text{h.c.} \Big] \\
& + \Big[\overline{\tilde{\chi}}_L^d \varepsilon_1^{d^\dagger} \; G_2(p, M_d, 2t_W\alpha + 2t_F\beta) \; \varepsilon_2^d \chi_R^d + \text{h.c.} \Big] \Big\} \; .
\end{aligned}
\tag{7.43}
$$

For the u quarks, one can proceed exactly in the same way. In the Euclidean space-time, the explicit expressions of the functions F, G_1 and G_2 are given by

$$
\begin{aligned}
F(p,M,\rho) &= \frac{1}{(\pi R)^2}\sum_{n=-\infty}^{\infty}\frac{1}{p^2+(\frac{n+\rho}{R})^2+M^2} \\
&= \frac{1}{\pi R\sqrt{p^2+M^2}}\,\mathrm{Re}\left[\sum_{m=-\infty}^{\infty} e^{-|2m|\pi R\sqrt{p^2+M^2}}e^{-|2m|\pi i\rho}\right] \\
&= \frac{1}{\pi R\sqrt{p^2+M^2}}\,\mathrm{Re}\coth(\pi R\sqrt{p^2+M^2}+i\rho\pi)\,, \\
G_1(p,M,\rho) &= \frac{1}{\pi R}\sum_{n=-\infty}^{\infty}(-1)^n\frac{\frac{n+\rho}{R}}{p^2+(\frac{n+\rho}{R})^2+M^2} \\
&= \mathrm{Im}\left[\sum_{m=-\infty}^{\infty} e^{-|2m+1|\pi R\sqrt{p^2+M^2}}e^{-|2m+1|\pi i\rho}\right] \\
&= -\mathrm{Im}\,\mathrm{csch}(\pi R\sqrt{p^2+M^2}+i\rho\pi)\,, \\
G_2(p,M,\rho) &= \frac{1}{\pi R}\sum_{n=-\infty}^{\infty}(-1)^n\frac{M}{p^2+(\frac{n+\rho}{R})^2+M^2} \\
&= \frac{M}{\sqrt{p^2+M^2}}\,\mathrm{Re}\left[\sum_{m=-\infty}^{\infty} e^{-|2m+1|\pi R\sqrt{p^2+M^2}}e^{-|2m+1|\pi i\rho}\right] \\
&= \mathrm{Re}\,\mathrm{csch}(\pi R\sqrt{p^2+M^2}+i\rho\pi)\,.
\end{aligned}
\tag{7.44}
$$

7.2.6 Low energy limit

We now study the effective Lagrangian in the low energy limit $p^2 \ll M_l^2$. In this limit, the non-local p-dependent couplings of eqs. (7.42)-(7.43) and the analogous terms for up-type quarks reduce to local kinetic and mass terms. After diagonalization and canonical normalization of the physical fields, these generate the physical fermion masses and mixings.

In the low-energy limit $p^2 \ll M_l^2$, the momentum variable $\pi R\sqrt{p^2+M_l^2}$ reduces to the constant parameter x_l defined in eq. (7.39). The functions F, G_1 and G_2 become simple trigonometric functions of the three parameters x_l, α and β. Notice moreover that not all the functional dependence on the parameters α and β is relevant. First of all, for various phenomenological reasons that were explained in ref. [22] and in sec. 7.1 and that we will review below, we must assume that α is small and retain only the leading effects that are at most linear in α. Since α is related to the VEV of the Higgs field, this corresponds to keeping only those effective operators that involve at most one Higgs field. Moreover, it is easy to check that only even powers of β are relevant in F and G_1, and similarly only odd powers of β are relevant in G_2, due to the flavour quantum numbers of the brane fields (see eqs. (7.12)-(7.15)). The above functions can therefore be effectively substituted with:

$$\begin{aligned}
F(p, M_l, 2t_W\alpha + 2t_F\beta) &\Rightarrow f(x_l, 2t_F\beta)\,, \\
G_1(p, M_l, 2t_W\alpha + 2t_F\beta) &\Rightarrow (2\pi t_W\alpha)\, g_1(x_l, 2t_F\beta)\,, \\
G_2(p, M_l, 2t_W\alpha + 2t_F\beta) &\Rightarrow (2\pi t_W\alpha)\, g_2(x_l, 2t_F\beta)\,,
\end{aligned} \tag{7.45}$$

where

$$\begin{aligned}
f(x_l, 2t_F\beta) &= \frac{1}{x_l}\,\mathrm{Re}\,\coth\Big[x_l + 2\pi i t_F\beta\Big]\,, \\
g_1(x_l, 2t_F\beta) &= \mathrm{Re}\,\Big(\coth\Big[x_l + 2\pi i t_F\beta\Big]\mathrm{csch}\Big[x_l + 2\pi i t_F\beta\Big]\Big)\,, \\
g_2(x_l, 2t_F\beta) &= \mathrm{Im}\,\Big(\coth\Big[x_l + 2\pi i t_F\beta\Big]\mathrm{csch}\Big[x_l + 2\pi i t_F\beta\Big]\Big)\,.
\end{aligned} \tag{7.46}$$

Let us be more quantitative on the range of values that the above dimensionless parameters are allowed to take by basic phenomenological constraints. A first important requirement is that $m_W \ll 1/R$, since indirect experimental constraints imply that the compactification scale should be at least a few TeV. A second requirement is that $M_l \gg m_W$, in such a way that even the lightest modes of the extra bulk fermions that we have introduced are heavy enough to satisfy direct experimental constraints. This two conditions imply respectively the following restrictions:

$$\pi\alpha \ll 1\,, \qquad \pi\alpha \ll x_l\,. \tag{7.47}$$

Notice that the above conditions justify in a more precise way the approximation done to derive eqs. (7.46). Notice also that they do not fix the size of the parameters x_l related to the masses of the bulk fermions.

The total effective Lagrangian is obtained by adding up the first rows of the brane Lagrangians (7.19)-(7.20) and the correction $\mathcal{L}_u^{\rm eff} + \mathcal{L}_d^{\rm eff}$. After simplifying the traces over gauge and flavour indices, which in the approximation leading to eqs. (7.46) are disentangled, it can be rewritten in terms of the original three generations of fields u_L, u_R, d_L, d_R and couplings $\epsilon^l_{L,R}$, and has the following general form:

$$\begin{aligned}
\mathcal{L}^{\rm phen} = \sum_{a,b=1}^{3} \Big\{ &\bar u_L^a \not p\, \mathcal{Z}_{ab}^{u_L} u_L^b + \bar u_R^a \not p\, \mathcal{Z}_{ab}^{u_R} u_R^b + \Big(\bar u_L^a \mathcal{M}_{ab}^u u_R^b + \mathrm{h.c.}\Big) \\
&+ \bar d_L^a \not p\, \mathcal{Z}_{ab}^{d_L} d_L^b + \bar d_R^a \not p\, \mathcal{Z}_{ab}^{d_R} d_R^b + \Big(\bar d_L^a \mathcal{M}_{ab}^d d_R^b + \mathrm{h.c.}\Big) \Big\}\,.
\end{aligned} \tag{7.48}$$

7.2.7 Fermion masses and mixings

Let us now specialize to the case at hand and work out in detail the expressions for fermion masses and mixing angles that can be obtained from $\mathcal{L}^{\rm phen}$ in eq. (7.48). In order to make the physics behind $\mathcal{L}^{\rm phen}$ clear, it is instructive to study the two limits $x_l \ll 1$ and $x_l \gg 1$, where many of the expressions drastically simplify. We start by discussing the case $x_l \gg 1$, since the corrections to the quark field wave functions are simpler in this limit.

$\mathbf{x_l \gg 1}$

In the limit of $x_l \gg 1$, the functions in eqs. (7.46) take the form

$$\begin{aligned} f(x_l, 2t_F\beta) &\sim \frac{1}{x_l}\left(1 + 2\,e^{-2x_l}\cos 4\pi t_F\beta\right) \sim \frac{1}{x_l}\,, \\ g_1(x_l, 2t_F\beta) &\sim 2e^{-x_l}\cos(2\pi t_F\beta)\,, \\ g_2(x_l, 2t_F\beta) &\sim -2e^{-x_l}\sin(2\pi t_F\beta)\,. \end{aligned} \tag{7.49}$$

Under the hypothesis that $\lambda = \pi\beta$ is of the order of the Cabibbo angle, we expand up to the appropriate order equations (7.49). For the case at hand, this order is λ^8. Our expansion gives the following effective mass matrices:

$$\mathcal{M}^d_{ad} = -\,m_W\;e^{-x_d}\,(\mathcal{E}^d_1)^\dagger_{ab}\,\widetilde{Y}^d_{bc}\,(\mathcal{E}^d_2)_{cd} \tag{7.50}$$

$$\mathcal{M}^u_{ad} = -\sqrt{2}\,m_W\;e^{-x_u}\,(\mathcal{E}^u_1)^\dagger_{ab}\,\widetilde{Y}^u_{bc}\,(\mathcal{E}^u_2)_{cd}\,, \tag{7.51}$$

where, keeping only the leading terms for each entry,

$$\widetilde{Y}^d = \begin{pmatrix} -2\sqrt{14}\,\lambda^5 & \sqrt{70}\,\lambda^4 & 2\sqrt{14}\,\lambda^3 \\ -5\sqrt{7}\,\lambda^4 & 2\sqrt{35}\,\lambda^3 & 3\sqrt{7}\,\lambda^2 \\ 10\,\lambda^2 & -2\sqrt{5}\,\lambda & -1 \end{pmatrix}, \tag{7.52}$$

$$\widetilde{Y}^u = \begin{pmatrix} \lambda^8 & -2\sqrt{14}\,\lambda^5 & 2\sqrt{14}\,\lambda^3 \\ 2\sqrt{2}\,\lambda^7 & -5\sqrt{7}\,\lambda^4 & 3\sqrt{7}\,\lambda^2 \\ -2\sqrt{14}\,\lambda^5 & 10\,\lambda^2 & -1 \end{pmatrix}, \tag{7.53}$$

and for convenience we have defined

$$\mathcal{E}^d_k = \mathrm{diag}(\epsilon^d_{k,1}, \epsilon^d_{k,2}, \epsilon^d_{k,3})\,, \qquad \mathcal{E}^u_k = \mathrm{diag}(\epsilon^u_{k,1}, \epsilon^u_{k,2}, \epsilon^u_{k,3})\,. \tag{7.54}$$

We see from eqs. (7.52) and (7.53) that we have obtained the desired structure in powers of λ, but the group-theoretical coefficients are large and modify substantially masses and mixing angles. However, since these coefficients are entirely fixed by the flavour symmetry, one can tolerate their presence and design the texture in such a way to obtain suitable additional powers of λ to compensate for the fact that they are not of order 1. In other words, we can still obtain a good description of masses and mixings in terms of a single parameter λ, but with non-conventional textures, which take into account the fact that the numerical coefficients can become of order λ^{-1} or larger. We will discuss in sec. 7.4.1 an explicit realization of this idea. The wave-function corrections are instead given by

$$
\begin{aligned}
\mathcal{Z}^{u_L} &= \mathcal{Z}^{d_L} = \mathbf{1} + \frac{1}{x_d}\mathcal{E}_1^{d\dagger}\mathcal{E}_1^d + \frac{1}{x_u}\mathcal{E}_1^{u\dagger}\mathcal{E}_1^u \,, \\
\mathcal{Z}^{d_R} &= \mathbf{1} + \frac{1}{x_d}\mathcal{E}_2^{d\dagger}\mathcal{E}_2^d \,, \\
\mathcal{Z}^{u_R} &= \mathbf{1} + \frac{1}{x_u}\mathcal{E}_2^{u\dagger}\mathcal{E}_2^u \,.
\end{aligned}
\tag{7.55}
$$

The physical quark Yukawa couplings are obtained by redefining the quark fields to reabsorb the wave-function corrections $\mathcal{Z}$. The structure of the latter is such that the physical mass matrix cannot grow indefinitely when the $\epsilon_a^{u,d}$ are increased. The reason is that the ϵ-parameters encode the mixing between bulk and brane fermions. The resulting mass of the hybrid fields must therefore interpolate between the value that one would get for a bulk field ($\epsilon_a^{u,d} \to \infty$) and the vanishing value that one would get for a brane field ($\epsilon_a^{u,d} \to 0$).

In the simple case where the ϵ_a^u and ϵ_a^d are real, it is useful to introduce the following bulk-brane mixing angles:

$$
\begin{aligned}
\alpha_{1,a}^u &= \mathrm{Arctan}\left(\frac{\sqrt{1/x_u}\,\epsilon_{1,a}^u}{\sqrt{1+1/x_d\,\left(\epsilon_{1,a}^d\right)^2}}\right)\,, \qquad &
\alpha_{1,a}^d &= \mathrm{Arctan}\left(\frac{\sqrt{1/x_d}\,\epsilon_{1,a}^d}{\sqrt{1+1/x_u\,\left(\epsilon_{1,a}^u\right)^2}}\right)\,, \\
\alpha_{2,a}^u &= \mathrm{Arctan}\left(\sqrt{1/x_u}\,\epsilon_{2,a}^u\right)\,, &
\alpha_{2,a}^d &= \mathrm{Arctan}\left(\sqrt{1/x_d}\,\epsilon_{2,a}^d\right)\,.
\end{aligned}
\tag{7.56}
$$

The physical masses, obtained by rescaling the quarks fields in order to have a canonically normalized kinetic term, e.g. $\bar{u}_L\,\not{p}\,u_L$, are then found to be:

$$
\begin{aligned}
\mathcal{M}^u &= -\sqrt{2}\,x_u\,e^{-x_u}\,m_W\,S_1^u\,\widetilde{Y}^u\,S_2^u\,, \\
\mathcal{M}^d &= -\,x_d\,e^{-x_d}\,m_W\,S_1^d\,\widetilde{Y}^d\,S_2^d\,,
\end{aligned}
\tag{7.57}
$$

where

$$
\begin{aligned}
S_1^l &= \mathrm{diag}(\sin\alpha_{1,1}^l, \sin\alpha_{1,2}^l, \sin\alpha_{1,3}^l)\,, \\
S_2^l &= \mathrm{diag}(\sin\alpha_{2,1}^l, \sin\alpha_{2,2}^l, \sin\alpha_{2,3}^l)\,.
\end{aligned}
\tag{7.58}
$$

At this point, we proceed exactly as in the SM, by diagonalizing the mass matrices via a bi-unitary transformation

$$u^{\alpha}_{L,R} \to \mathcal{U}^{\alpha\beta}_{L,R} u^{\beta}_{L,R}\,, \quad d^{\alpha}_{L,R} \to \mathcal{D}^{\alpha\beta}_{L,R} d^{\beta}_{L,R} \;\;\Rightarrow\;\; V_{CKM} = \mathcal{U}^{\dagger}_{L}\mathcal{D}_{L}\,. \tag{7.59}$$

The masses in eq. (7.57) are suppressed with respect to $m_W = \alpha/R$ by the factor $x_l e^{-x_l}$, which is a small parameter since we are now considering the limit $x_l \gg 1$, and by a trigonometric factor parametrizing the bulk-brane mixing. In this situation we therefore obtain mass matrices with an absolute scale much smaller than the W mass:

$$\mathcal{M}^{u,d} \sim x_{u,d} e^{-x_{u,d}} m_W \ll m_W\,. \tag{7.60}$$

This is phenomenologically not acceptable for the top quark mass. Notice, nevertheless, that the exponential dependence on x_u and x_d of the overall scale for the masses in the up and down sectors could allow to account for the significant hierarchy observed between the latter through a modest hierarchy between the two parameters x_u and x_d. The physical origin of the above exponential suppression is related to the higher-dimensional gauge symmetry constraining the Higgs interactions. More precisely, the only invariant Yukawa-type effective operators turn out to involve the Higgs field in the form of a Wilson line, which connects the two branes where the relevant left- and right-handed fermions are located and winds at least once around the internal interval [22]. The exchanged bulk fermion of mass M_l must therefore propagate at least over a distance πR and this implies a suppression factor proportional to e^{-x_l} in the limit $x_l \gg 1$.

$\mathbf{x_l \ll 1}$

In the limit of $x_l \ll 1$, the functions in eqs. (7.46) also simplify. Actually, to have a significant simplification we really need $x_l \ll \pi\beta$, but deriving an asymptotic expression in this limit would contrast with the philosophy of flavour models, which always assumes a power expansion in the order parameter $\pi\beta \ll 1$. For this reason, we will consider this situation only for the case of flavour-singlet bulk fermions, which are blind to the flavour symmetry. We will see in sec. 7.4.2 that it is possible to take advantage of the possibility of adding such a flavour-neutral fermion, in addition to flavour-charged ones, to improve the magnitude of the masses of the third family of quarks. We therefore set $\pi t_F \beta$ to 0. Under these assumptions, the functions of eqs. (7.46) reduce to

$$f(x_l, 0) \simeq \frac{1}{x_l^2}\,, \qquad g_1(x_l, 0) \simeq \frac{1}{x_l^2}\,, \qquad g_2(x_l, 0) \simeq 0\,. \tag{7.61}$$

The induced wave functions are then given by (there is no matrix structure here since we are considering flavour singlets):

$$\mathcal{Z}^l_L \simeq 1 + \frac{1}{x_d^2}\epsilon^{d\dagger}_L \epsilon^d_L + \frac{1}{x_u^2}\epsilon^{u\dagger}_L \epsilon^u_L\,, \qquad \mathcal{Z}^l_R \simeq 1 + \frac{1}{x_l^2}\epsilon^{l\dagger}_R \epsilon^l_R\,. \tag{7.62}$$

Similarly, the induced masses are found to be

$$\mathcal{M}^u \simeq \sqrt{2}\frac{1}{x_u^2}\epsilon_L^{u\dagger}\epsilon_R^u m_W\,, \qquad \mathcal{M}^d \simeq \frac{1}{x_d^2}\epsilon_L^{d\dagger}\epsilon_R^d m_W\,. \tag{7.63}$$

The physical quark masses emerging after canonical normalization are then found to be

$$m^u \simeq \sqrt{2}\sin\alpha_L^u \sin\alpha_R^u\, m_W\,, \qquad m^d \simeq \sin\alpha_L^d \sin\alpha_R^d\, m_W\,, \tag{7.64}$$

where now

$$\begin{aligned} &\alpha_L^u = \arctan\sqrt{\frac{(\epsilon_L^u)^2/x_u^2}{1+(\epsilon_L^d)^2/x_d^2}}\,, \qquad &&\alpha_L^d = \arctan\sqrt{\frac{(\epsilon_L^d)^2/x_d^2}{1+(\epsilon_L^u)^2/x_u^2}}\,, \\ &\alpha_R^u = \arctan\sqrt{(\epsilon_R^u)^2/x_u^2}\,, &&\alpha_R^d = \arctan\sqrt{(\epsilon_R^d)^2/x_d^2}\,. \end{aligned} \tag{7.65}$$

In this case the quark masses are of order m_W. In this situation we can therefore achieve mass matrices with a trivial flavour structure but a sizable magnitude:

$$\frac{m^l}{m_W} \sim 1\,. \tag{7.66}$$

Notice also that for $\epsilon_{L,R}^l \sim 1$ the angles (7.65) parametrizing the brane-bulk mixings tend to the large values $\alpha_L^u \simeq \delta$, $\alpha_L^d \simeq \pi/2 - \delta$ and $\alpha_R^l \simeq \pi/2$, with $\delta = \mathrm{Arctan}(\epsilon_L^u/\epsilon_L^d x^d/x^u)$, reflecting the fact that since $\epsilon_{L,R}^l \gg x_l$ the brane-bulk mixing is maximal; the masses (7.64) tend then to $m^d \simeq \cos\delta\, m_W$ and $m^u \simeq \sqrt{2}\sin\delta\, m_W$.

$x_l \sim 1$

In the general case $x_l \sim 1$, the effect of the wave-function corrections on the $\mathcal{O}(1)$ numerical coefficients in the physical Yukawa couplings depends in a complicated way on the parameters x_l and $\epsilon_{1,2}^l$, and must be separately studied for each point in this parameter space. We will present the results of this general analysis in sec. 7.4. It is however clear that the induced masses will always have a scale that is parametrically given by m_W times some suppression factor dictated by the spontaneously broken flavor symmetry. As already mentioned in the introduction, the large top mass is therefore generically difficult to accommodate in this framework [22].

7.3 Generalization to arbitrary representations

In this section, we generalize the construction discussed above to arbitrary representations of the electroweak and flavour groups.

We generalize the minimal choice of ref. [22] by taking ψ^d and ψ^u to belong respectively to the $(\mathbf{n_W^d}+\mathbf{1})(\mathbf{n_W^d}+\mathbf{2})/\mathbf{2}$ (n_W^d times symmetric) and $(\mathbf{n_W^u}+\mathbf{1})(\mathbf{n_W^u}+\mathbf{2})/\mathbf{2}$ (n_W^u times symmetric) of $SU(3)_W$; the $\mathbf{3}$ and $\mathbf{6}$ that were used in ref. [22] and in the previous discussion correspond to the particular cases $n_W^d = 1$ and $n_W^u = 2$. Moreover, we take these fields to belong to the $\mathbf{2j_F}+\mathbf{1}$ (spin-j_F) representation of $SU(2)_F$, so that there are now $2j_F+1$ replicas of them with identical $SU(2)_L \times U(1)_Y$ quantum numbers but different $U(1)_F$ charges. Summarizing, we have bulk fields in the following representations of $SU(3)_W \times SU(2)_F$:

$$\psi^l, \tilde{\psi}^l : \Big(\frac{(\mathbf{n_W^l}+\mathbf{1})(\mathbf{n_W^l}+\mathbf{2})}{\mathbf{2}}, \mathbf{2j_F}+\mathbf{1}\Big)\,, \tag{7.67}$$

The decomposition of the above general representations of the $SU(3)_W \times SU(2)_F$ group under its $SU(2)_L \times U(1)_Y \times U(1)_F$ subgroup, which we need to determine the coupling of the bulk fields to the brane fields, has the following form:

$$\Big(\frac{(\mathbf{n_W^l}+\mathbf{1})(\mathbf{n_W^l}+\mathbf{2})}{\mathbf{2}}, \mathbf{2j_F}+\mathbf{1}\Big) \rightarrow \bigoplus_{j_W=0}^{n_W^l/2} \bigoplus_{m_{j_F}=-j_F}^{j_F} (\mathbf{2j_W}+\mathbf{1})_{j_W - n_W^l/3, m_{j_F}}\,. \tag{7.68}$$

We get therefore a set of representations of $SU(2)_L$ with half-integer spins j_W ranging from 0 to $n_W^l/2$, canonically normalized $U(1)_Y$ charge equal to $j_W - n_W^l/3$ and $U(1)_F$ charges m_{j_F} ranging from $-j_F$ to j_F. The only components that have the right quantum numbers to couple to the brane fermions are the $SU(2)_L$ doublets and singlets with $j_W = 1/2$ and $j_W = 0$, which have $U(1)_Y$ charge[3] equal to $1/2 - n_W^l/3$ and $-n_W^l/3$, and $U(1)_F$ charges ranging from $-j_F$ to j_F.

The action of the orbifold projection and the SS twist on the bulk fermion fields can be easily deduced by using some simple group-theoretical techniques. In the electroweak sector, the completely symmetric representations of $SU(3)_W$ we are considering contain states with values of the two Cartan generators T_W^3 and $2\,T_W^8/\sqrt{3}$ that fill an equilateral triangle in the corresponding plane. This triangle is oriented with its tip at the bottom and one of his sides at the top and horizontal. It can be sliced in essentially two different ways in a sum of lines, corresponding to decompositions with respect to nonequivalent but isomorphic maximal subgroups. Slicing the $SU(3)_W$ representation horizontally in rows, one obtains the decomposition with respect to the $SU(2)_L \times U(1)_Y$ preserved by the

[3]Notice that these have automatically the right hypercharge to couple to the standard left-handed doublets and right-handed singlets only in the special case $n_W^d = 1$ and $n_W^u = 2$ chosen in ref. [22]. For more general values of $n_W^d \neq 1$ and $n_W^u \neq 2$, one needs to assign to the bulk fields a non-vanishing charge under the extra $U(1)'$ factor that is needed to tune the weak mixing angle, which is equal to $(n_W^d - 1)/3$ for $\psi^d, \tilde{\psi}^d$ and $(n_W^u - 2)/3$ for $\psi^u, \tilde{\psi}^u$. Notice however that unless these charges are opposite to each other, that is if $n_W^u + n_W^d = 3$, two different fields are needed to give mass to the u and the d quarks, due to the restrictions set by the $U(1)'$-invariance of the coupling to the left-handed quarks.

orbifold projection, with generators $T_W^{1,2,3}$ and $T_W^8/\sqrt{3}$. It is then clear that the generator $T_W^8/\sqrt{3}$ appearing in the orbifold projection has a definite value for each $SU(2)_L \times U(1)_Y$ representation appearing in the decomposition (7.68). More precisely, it acts as $j_W - n_W^l/3$ on the component with $SU(2)_L$ spin j_W. In matrix form, where these components are ordered in block with a fixed j_W ranging from $n_W^l/2$ to 0 in decreasing order and subentries corresponding to m_{j_W} ranging from $-j_W$ to j_W in increasing order[4], the orbifold twist has therefore the following form:

$$P_W^{\mathbf{(n^l_W+1)(n^l_W+2)/2}} = \mathrm{diag}(\underbrace{(-1)^{n_W^l}, \ldots, (-1)^{n_W^l}}_{n_W^l+1 \ \text{times}}; \ldots; 1, 1, 1; -1, -1; 1)\,. \tag{7.69}$$

Slicing the $SU(3)_W$ representation diagonally, that is parallel to one of the two non-horizontal sides of the triangle, one obtains the decomposition with respect to a different $SU(2)' \times U(1)'$ subgroup associated to the Scherk-Schwarz twist, with generators $T_W^{6,7}$, $(-T_W^3 + \sqrt{3}\,T_W^8)/2$ and $(-T_W^3 - T_W^8/\sqrt{3})/2$. For each state of the original representation, the $SU(2)'$ spin j' and its third component $m_{j'}$ are related to the the $SU(2)_L$ spin j_W and its third component m_{j_W} by the relations $j' = (n_W^l - j_W - m_{j_W})/2$ and $m_{j'} = (-n_W^l + 3j_W - m_{j_W})/2$. This decomposition is useful to determine the action of the generator T_W^6 appearing in the Scherk-Schwarz twist. Indeed, one can rewrite $T_W^6 = (T_W^+ + T_W^-)/2$ in terms of the raising and lowering operators $T_W^\pm = T_W^6 \pm i T_W^7$ of the $SU(2)'$ subgroup. These leave j' unchanged and raise/lower $m_{j'}$ by 1 unit, or equivalently, they raise/lower j_W by 1/2 unit and lower/raise m_{j_W} by 1/2 unit. The generator T_W^6 acts in a non-diagonal way on the decomposition (7.68), but its matrix elements can be easily determined using the standard SU(2) results. Its diagonal form is also easily derived, thanks to the fact that any generator of an SU(2) group has the same diagonal form, due to the fact that there is only one Cartan generator. In our case, the diagonal form of T_W^6 must coincide in from with the generator $(-T_W^3 + \sqrt{3}\,T_W^8)/2$ representing the third component of the $SU(2)'$ spin. In terms of the quantum numbers defined by the decomposition (7.68), the latter acts as $(-n_W^l + 3j_W - m_{j_W})/2$ on the m_{j_W}-th element of the spin j_W component. In the same matrix notation as above, this means

$$t_W^{\mathbf{(n^l_W+1)(n^l_W+2)/2}} = \mathrm{diag}(0, \frac{1}{2}, \ldots, \frac{n_W^l}{2}; \ldots; -\frac{n_W^l}{2} + \frac{1}{2}, -\frac{n_W^l}{2} + 1; -\frac{n_W^l}{2})\,. \tag{7.70}$$

In the flavour sector, the situation is similar but much simpler, since we start with an $SU(2)_F$ group. The generator T_F^3 appearing in the orbifold projection is just the third component of the $SU(2)_F$ spin, and acts therefore as m_{j_F} on the m_{j_F}-th component of the decomposition (7.68). One then finds that the projection matrix P_F acts as $(-1)^{j_F - m_{j_F}}$ on the m_{j_F}-th component of the representation. In matrix notation, where these components

[4]This ordering of the states differs from the one used for the particular example of section 3.

are ordered with decreasing m_{j_F} ranging from j_F to $-j_F$[5], the orbifold twist has therefore the following form:

$$P_F^{\mathbf{2j_F+1}} = \mathrm{diag}(1,-1,1,-1,\dots)\,. \tag{7.71}$$

The generator T_F^1 appearing in the Scherk-Schwarz twist can be written more usefully as $T_F^1 = (T_F^+ + T_F^-)/2$ in terms of the raising and lowering operators $T_W^{\pm} = T_F^1 \pm iT_W^2$ of the $SU(2)_F$ subgroup. This allows to compute in a simple way any of its matrix elements. Its diagonal form must coincide with that of the Cartan generator T_F^3, which acts as m_{j_F} on the m_{j_F}-th component of the decomposition (7.68). The diagonal form of the twist is therefore given by

$$t_F^{\mathbf{2j_F+1}} = \mathrm{diag}\Big(j_F, j_F-1, \dots, -j_F+1, -j_F\Big)\,. \tag{7.72}$$

We now describe the general situation that can be achieved in this more generic setting, in order to illustrate the basic features of the construction and its peculiarities compared to standard 4D flavour models.

7.3.1 Lagrangian

The structure of the Lagrangian is the same as in the previous section. The couplings of the family triplets of left- and right-handed brane fields $\phi = Q_L, u_R, d_R$ and their conjugates $\phi^c = Q_R^c, \text{-}u_L^c, d_L^c$ to the bulk fields ψ^l or $\tilde\psi^l$ are parametrized by family triplets of couplings e_1^l and e_2^l with mass-dimension 1/2, in each sector $l = u, d$. Each ϕ or ϕ^c can couple either to ψ^l or $\tilde\psi^l$, and has therefore only one relevant coupling. To write these couplings more explicitly, it is convenient to embed the fields ϕ and ϕ^c into new fields $\Phi = Q, u, d, \tilde Q, \tilde u, \tilde d$ and their conjugates $\Phi^c = Q^c, u^c, d^c, \tilde Q^c, \tilde u^c, \tilde d^c$, which have the same matrix structure as the representations of $SU(3)_W \times SU(2)_F$ to which the bulk fields they couple to belong, the extra entries being filled with zeroes[6]. The untilded and tilded fields in Φ or Φ^c contain those SM fermions ϕ or ϕ^c that have the right quantum numbers to couple to ψ^l and $\tilde\psi^l$ respectively. With this notation, which is the appropriate generalization of the one used to deal with the particular example of sec. 3, the Lagrangian is obtained from eq. (7.17) by replacing the localized terms with

$$\begin{aligned}\mathcal{L}^0 &= i\bar Q\gamma^\mu D_\mu Q + i\bar{\tilde Q}\gamma^\mu D_\mu \tilde Q \\ &\quad + \Big[\bar Q\,\hat e_1^{d\dagger}\psi^d + \bar{\tilde Q}\,\hat e_1^{d\dagger}\tilde\psi^d + \bar Q^c\,\hat e_1^{u\dagger}\psi^u + \bar{\tilde Q}^c\,\hat e_1^{u\dagger}\tilde\psi^u + \mathrm{h.c.}\Big]\,,\end{aligned} \tag{7.73}$$

$$\begin{aligned}\mathcal{L}^{\pi R} &= i\bar u^c\gamma^\mu D_\mu u^c + i\bar{\tilde u}^c\gamma^\mu D_\mu \tilde u^c + i\bar{\tilde d}\gamma^\mu D_\mu \tilde d + i\bar d\gamma^\mu D_\mu d \\ &\quad + \Big[\bar d\,\hat e_2^{d\dagger}\psi^d + \bar{\tilde d}\,\hat e_2^{d\dagger}\tilde\psi^d + \bar u^c\,\hat e_2^{u\dagger}\psi^u + \bar{\tilde u}^c\,\hat e_2^{u\dagger}\tilde\psi^u + \mathrm{h.c.}\Big]\,.\end{aligned} \tag{7.74}$$

[5]Again, this ordering differs from the canonical one used for the particular example of section 3.

[6]We denote the new embedded fields with the same letter as the original ones, but drop the L, R subscripts to them.

To be more precise about the embeddings, let us denote $SU(2)_L \times U(1)_Y$ and family indices by $\alpha, \beta, \dots$ and $I, J, \dots = 1, 2, 3$, and $SU(3)_W$ and $SU(2)_F$ indices by $i, j, \dots$ and $a, b, \dots$. The embedding of each field is then specified by some $(n^l_W + 1)(n^l_W + 2)/2$ by $2j_W + 1$ matrix $(\mathcal{I}_W)_{i\alpha}$ for gauge indices, where j_W is 0 for singlets and 1/2 for doublets, and similarly by some $2j_F + 1$ by 3 matrix $(\mathcal{I}_F)_{aI}$ for flavour indices. The position of each field ϕ or ϕ^c in Φ or Φ^c is uniquely determined by its $SU(2)_L \times U(1)_Y$ and $U(1)_F$ quantum numbers in the gauge and flavour sectors respectively. For the couplings, the embedding is trivial for gauge indices and is determined in an obvious way in terms of that of the fields for flavour indices: it is a diagonal $2j_F + 1$ by $2j_F + 1$ matrix whose non-zero entries are the couplings that are relevant for each field, in the corresponding positions.

The embedding in the gauge sector generalizes the one used in ref. [22]. Rather than reporting the matrices $\mathcal{I}_W$ for each field, we can exhibit the same information by reporting the expressions of the fields $\Phi_W = \mathcal{I}^{\Phi}_W \phi$ and $\Phi^c_W = \mathcal{I}^{\Phi^c}_W \phi^c$. These are $(n^l_W + 1)(n^l_W + 2)/2$-dimensional vectors will all the entries set to zero apart from the last three, which host the SM fields:

$$Q_W = \tilde{Q}_W = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u_L \\ d_L \\ 0 \end{pmatrix} , \qquad d_W = \tilde{d}_W = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ d_R \end{pmatrix} , \tag{7.75}$$

$$Q^c_W = \tilde{Q}^c_W = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ d^c_R \\ \text{-}u^c_R \\ 0 \end{pmatrix} , \qquad u^c_W = \tilde{u}^c_W = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \text{-}u^c_L \end{pmatrix} . \tag{7.76}$$

The embedding in the flavour sector is done in a similar way and depends on the choice of flavour quantum numbers. Again, rather than reporting the matrices $\mathcal{I}_F$ for each field, one can consider directly the redefined fields $\Phi_F = \mathcal{I}^{\Phi}_F \phi$ and $\Phi^c_F = \mathcal{I}^{\Phi^c}_F \phi^c$. For Φ_F, each SM fermion ϕ is embedded at the $(j_F - q_F + 1)$-th entry if its flavour charge is q_F, and appears only in the untilded or tilded redefined fields if $j_F - q_F$ is respectively even or odd. Similarly, for the conjugate Φ^c_F, each conjugate SM fermion ϕ^c is embedded at the $(j_F + q_F + 1)$-th entry if its flavour charge is $-q_F$, and appears only in the untilded or tilded redefined fields if $j_F - q_F$ is respectively even or odd. As a consequence, for the embedding of the SM fields ϕ in Φ, only the odd and even entries of respectively the untilded and the tilded redefined fields are relevant, all the other being always zero; for the embedding of the conjugate SM fields ϕ^c in Φ^c, the situation is similar, and Φ^c is obtained from Φ through a reflection. Schematically, the structure is as follows, with at most three non-vanishing entries for each vector:

$$Q_F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (Q_L)_{I_1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} , \quad \tilde{Q}_F = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (Q_L)_{\tilde{I}_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} , \quad d_F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (d_L)_{J_1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} , \quad \tilde{d}_F = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (d_L)_{\tilde{J}_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} , \tag{7.77}$$

$$Q^c_F = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (Q^c_R)_{I_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} , \quad \tilde{Q}^c_F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (Q^c_R)_{\tilde{I}_1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} , \quad u^c_F = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (\text{-}u^c_L)_{K_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} , \quad \tilde{u}^c_F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (\text{-}u^c_L)_{\tilde{K}_1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} . \tag{7.78}$$

In this expressions, $I_1,\ J_1,\ K_1$ and $\tilde{I}_1,\ \tilde{J}_1,\ \tilde{K}_1$ are restricted family indices running respectively over those families for which the left-handed doublets, the right-handed down singlets and the right-handed up singlets are embedded in untilded and tilded vectors.

Finally, the brane-bulk couplings are correspondingly embedded into diagonal matrices $\hat{e}^l_{1,2}$ at those entries that correspond to a non-vanishing entry of the redefined fields. They have the following schematic form, with three non-vanishing entries labeled by a family index M:

$$\hat{e}^l_{1,2} = \text{diag}(0,\ldots,0,(e^l_{1,2})_{M_1},0,\ldots,0,(e^l_{1,2})_{M_2},0,\ldots,0,(e^l_{1,2})_{M_3},0,\ldots,0)\,. \tag{7.79}$$

7.3.2 Fermion masses and mixings

From the Lagrangian above one can proceed exactly as in Sec. 7.2 to derive the effective Lagrangian for the SM fermions, which still has the form of eq. (7.48). The general expressions of $\mathcal{M}$ and $\mathcal{Z}$ depend on the matrix elements of the generator T^6_W implementing the electroweak symmetry breaking and those of arbitrary powers of the generator T^1_F implementing the flavour symmetry breaking, which appear in the functions of eqs. (7.45). The relevant matrix element of T^6_W is universal and can be computed in general. It is the one connecting the next-to-last element of the embedding vector of the left-handed fields and their conjugates, that is the $m_{j_W} = -1/2$ component of the doublet with $j_W = 1/2$, and the last element of the embedding vector of the right-handed fields and their conjugates, that is the singlet with $m_{j_W} = 0$ and $j_W = 0$. As already explained, this can be easily evaluated by rewriting $T_W^6 = (T_W^+ + T_W^-)/2$ in terms of the raising and lowering operators $T_W^{\pm} = T_W^6 \pm i T_W^7$ of the $SU(2)'$ subgroup defined by the twist, which have non-vanishing matrix elements between neighbour states, namely $\sqrt{(j' \mp m_{j'})(j' \pm m_{j'} + 1)}$. The matrix element we are interested in is therefore an ordinary transition from the component with $m_{j'} = -n_W^l/2$ to the component with $m_{j'} = -n_W^l/2 + 1$ of an $SU(2)'$ representation of spin $j' = n_W^l/2$, and gives a factor $\sqrt{n_W^l}/2$. The matrix elements of a generic power of T_F^1 can be computed similarly. Here we simply rewrite the flavour traces in terms of the $2j_F + 1$ by 3 matrices $\mathcal{I}_F$ defining how the family triplet of each SM field is embedded into an $(2j_F + 1)$-dimensional flavour vector. The results are given by the following expressions:

$$
\begin{aligned}
\mathcal{Z}_L^d &= \mathbf{1} + \mathcal{E}_1^{d\dagger} \Big[\mathcal{I}_F^{Q\dagger} f(x_d, T_F^1 \beta)\, \mathcal{I}_F^{Q} + \mathcal{I}_F^{\tilde{Q}\dagger} f(x_d, T_F^1 \beta)\, \mathcal{I}_F^{\tilde{Q}} \Big] \mathcal{E}_1^d \\
&\quad + \mathcal{E}_1^{u\dagger} \Big[\mathcal{I}_F^{Q^c\dagger} f(x_u, T_F^1 \beta)\, \mathcal{I}_F^{Q^c} + \mathcal{I}_F^{\tilde{Q}^c\dagger} f(x_u, T_F^1 \beta)\, \mathcal{I}_F^{\tilde{Q}^c} \Big] \mathcal{E}_1^u \,, \\
\mathcal{Z}_L^u &= \mathbf{1} + \mathcal{E}_1^{u\dagger} \Big[\mathcal{I}_F^{Q^c\dagger} f(x_u, T_F^1 \beta)\, \mathcal{I}_F^{Q^c} + \mathcal{I}_F^{\tilde{Q}^c\dagger} f(x_u, T_F^1 \beta)\, \mathcal{I}_F^{\tilde{Q}^c} \Big] \mathcal{E}_1^u \\
&\quad + \mathcal{E}_1^{d\dagger} \Big[\mathcal{I}_F^{Q\dagger} f(x_d, T_F^1 \beta)\, \mathcal{I}_F^{Q} + \mathcal{I}_F^{\tilde{Q}\dagger} f(x_d, T_F^1 \beta)\, \mathcal{I}_F^{\tilde{Q}} \Big] \mathcal{E}_1^d \,, \\
\mathcal{Z}_R^d &= \mathbf{1} + \mathcal{E}_2^{d\dagger} \Big[\mathcal{I}_F^{d\dagger} f(x_d, T_F^1 \beta)\, \mathcal{I}_F^{d} + \mathcal{I}_F^{\tilde{d}\dagger} f(x_d, T_F^1 \beta)\, \mathcal{I}_F^{\tilde{d}} \Big] \mathcal{E}_2^d \,, \\
\mathcal{Z}_R^u &= \mathbf{1} + \mathcal{E}_2^{u\dagger} \Big[\mathcal{I}_F^{u^c\dagger} f(x_u, T_F^1 \beta)\, \mathcal{I}_F^{u^c} + \mathcal{I}_F^{\tilde{u}^c\dagger} f(x_u, T_F^1 \beta)\, \mathcal{I}_F^{\tilde{u}^c} \Big] \mathcal{E}_2^u \,,
\end{aligned}
\tag{7.80}
$$

and

$$
\begin{aligned}
\mathcal{M}^d &= \sqrt{n_W^d}\, \mathcal{E}_1^{d\dagger} \Big[\mathcal{I}_F^{Q\dagger} g_1(x_d, T_F^1 \beta)\, \mathcal{I}_F^{d} - \mathcal{I}_F^{\tilde{Q}\dagger} g_1(x_d, T_F^1 \beta)\, \mathcal{I}_F^{\tilde{d}} \\
&\qquad + \mathcal{I}_F^{Q\dagger} g_2(x_d, T_F^1 \beta)\, \mathcal{I}_F^{\tilde{d}} + \mathcal{I}_F^{\tilde{Q}\dagger} g_2(x_d, T_F^1 \beta)\, \mathcal{I}_F^{d} \Big] \mathcal{E}_2^d\, m_W \,, \\
\mathcal{M}^u &= \sqrt{n_W^u}\, \mathcal{E}_1^{u\dagger} \Big[\mathcal{I}_F^{Q^c\dagger} g_1(x_u, T_F^1 \beta)\, \mathcal{I}_F^{u^c} - \mathcal{I}_F^{\tilde{Q}^c\dagger} g_1(x_u, T_F^1 \beta)\, \mathcal{I}_F^{\tilde{u}^c} \\
&\qquad + \mathcal{I}_F^{Q^c\dagger} g_2(x_u, T_F^1 \beta)\, \mathcal{I}_F^{\tilde{u}^c} + \mathcal{I}_F^{\tilde{Q}^c\dagger} g_2(x_u, T_F^1 \beta)\, \mathcal{I}_F^{u^c} \Big] \mathcal{E}_2^u\, m_W \,.
\end{aligned}
\tag{7.81}
$$

Once the above quantities have been computed, the physical implications of the Lagrangian (7.48) are uniquely determined and can be analyzed as follows. First, one performs a suitable redefinition of the fermions fields to reabsorb the non-trivial wave function factor and canonically normalize their kinetic terms. In this process, the mass matrices will however be changed to new matrices $\hat{\mathcal{M}}^l$. Second, one proceeds as in the

SM and diagonalizes these two mass matrices through some unitary transformations $\mathcal{U}_{L,R}$ and $\mathcal{D}_{L,R}$ in the u and d sectors. This will then induce a CKM mixing matrix given by $V_{CKM} = \mathcal{U}_L^\dagger \mathcal{D}_L$.

The above procedure is complicated by the non-diagonal field redefinition that is required to get read of the wave function. One might fear that the new mass matrices $\hat{\mathcal{M}}^l$ that are generated after wave-function renormalization might have hierarchical structures in powers of λ that are messed up compared to those of $\mathcal{M}^l$. However, as shown in general in Sec. 7.2.4, this is not the case: at most the order one coefficients multiplying the powers of λ in the various entries are changed. We now generalize the discussion of Sec. 7.2.7 for generic representations.

$\mathbf{x_l \gg 1}$

In the limit of $x_l \gg 1$, the functions in eqs. (7.46) simplify to the form in eq. (7.49). As we have seen, at leading order in e^{-x_l} the wave functions reduce to diagonal constants:

$$\mathcal{Z}_L^l \simeq \mathbf{1} + \frac{1}{x_d}\mathcal{E}_L^{d\dagger}\mathcal{E}_L^d + \frac{1}{x_u}\mathcal{E}_L^{u\dagger}\mathcal{E}_L^u \,, \qquad \mathcal{Z}_R^l \simeq \mathbf{1} + \frac{1}{x_l}\mathcal{E}_R^{l\dagger}\mathcal{E}_R^l \,. \tag{7.82}$$

The masses $\mathcal{M}^l$ take instead the form

$$\mathcal{M}^l \simeq \sqrt{n_W^l}\, e^{-x_l}\mathcal{E}_L^{l\,\dagger}\widetilde{Y}^l\mathcal{E}_R^l\, m_W \,, \tag{7.83}$$

where $\widetilde{Y}^l$ are two 3×3 matrices that are functions of λ and carry all the information about the group-theoretical details of the flavour sector. Assuming that $\lambda \ll 1$, they have the form (7.21), but with completely fixed numerical coefficients, which can be easily computed using the standard realization of the $SU(2)$ algebra in terms of raising and lowering operators. Further assuming, for simplicity and without loss of generality, that the flavour charges l_I of the right-handed fields are larger than the charges q_I of the left-handed fields, and recalling that $\lambda = \pi\beta$, the result is, modulo a sign:

$$\widetilde{Y}_{IJ}^l \simeq \prod_{k=1}^{l_J-q_I}\sqrt{1+\frac{j_F-l_J}{k}}\sqrt{1+\frac{j_F+q_I}{k}}\,\lambda^{l_J-q_I} \,. \tag{7.84}$$

The first subleading corrections to these expressions can be easily evaluated using again creation and annihilation operators. The relative effect represented by these corrections is of order λ^2, and its precise expression, modulo a sign, is given by

$$\frac{\Delta\widetilde{Y}_{IJ}^l}{\widetilde{Y}_{IJ}^l} \simeq \sum_{k=0}^{l_J-q_I+1}\frac{(j_F+q_I+k)(j_F-q_I-k+1)}{(l_J-q_I+1)(l_J-q_I+2)}\,\lambda^2 \,. \tag{7.85}$$

From this expression it is clear that there is an obstruction against increasing too much the spin j_F of the representation of the bulk mediators at fixed flavour charges for the brane fields. Indeed, doing so increases the relative impact of the subleading terms and puts a limit on how large the parameter λ can be at fixed j_F, or viceversa how large j_F can be at fixed λ, without spoiling the simple idea that the Yukawa texture is fixed by the leading terms with powers of λ fixed by the charges. Notice for instance that in the extreme limit in which j_F is much larger than all of the charges, one finds that the leading term (7.84) goes like $j_F/(l_J - q_I)!\,\lambda^{l_J - q_I}$ if $l_J \neq q_I$ and 1 if $l_J = q_I$, whereas the relative subleading correction (7.85) goes like $j_F^2/(l_J - q_I + 1)\,\lambda^2$. Requiring that the latter be much smaller than 1 then implies that $j_F \ll \sqrt{l_J - q_I + 1}/\lambda$. For $\lambda \sim 10^{-1}$ and reasonable charges, one must then take $j_F \ll 10$. For $j_F \sim 3 - 4$, as in the examples that we shall study below, the subleading corrections represent therefore a significant error of about 10%.

The physical quark Yukawa couplings are obtained by redefining the quark fields to reabsorb the wave-function corrections $\mathcal{Z}^l_{L,R}$. The physical mass matrices are then found to be (see eq. (7.57)):

$$m^l \simeq \sqrt{n^l_W}\, x_l e^{-x_l} \sin\alpha^l_L\, \widetilde{Y}^l \sin\alpha^l_R\, m_W\,. \tag{7.86}$$

$\mathbf{x_l \ll 1}$

In the limit of $x_l \ll 1$, the masses in eq. (7.63) generalize as

$$\mathcal{M}^l \simeq \sqrt{n^l_W}\frac{1}{x_l^2}\epsilon^{l\dagger}_L \epsilon^l_R\, m_W\,. \tag{7.87}$$

The physical quark masses emerging after canonical normalization are then found to be

$$m^l \simeq \sqrt{n^l_W}\sin\alpha^l_L \sin\alpha^l_R\, m_W\,. \tag{7.88}$$

7.4 Model building

In this section, we apply the general construction developed so far to build viable flavour models. We present two illustrative examples that emphasize some important phenomenological aspects.

7.4.1 Mixing angles and mass ratios

The model presented in sec. 7.2 produces the correct structure of powers of λ for Yukawa couplings, but suffers from large group-theoretical coefficients that spoil the success of the chosen texture. The simplest way to solve this problem is to assign charges in such a way as no $\mathcal{O}(\lambda^0)$ term is present. Then, all entries will have comparable numerical coefficients, and the power expansion will be consistent. We can start for instance from

$$Y^d \sim \begin{pmatrix} \lambda^6 & \lambda^5 & \lambda^4 \\ \lambda^5 & \lambda^4 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda \end{pmatrix}, \qquad Y^u \sim \begin{pmatrix} \lambda^7 & \lambda^6 & \lambda^4 \\ \lambda^6 & \lambda^5 & \lambda^3 \\ \lambda^4 & \lambda^3 & \lambda \end{pmatrix}. \tag{7.89}$$

The simplest flavour charge assignment for the brane fermions that is compatible with these textures is given by

$$q_I = \left\{-\frac{7}{2}, -\frac{5}{2}, -\frac{1}{2}\right\}, \qquad d_I = \left\{\frac{5}{2}, \frac{3}{2}, \frac{1}{2}\right\}, \qquad u_I = \left\{\frac{7}{2}, \frac{5}{2}, \frac{1}{2}\right\}. \tag{7.90}$$

Since the maximal absolute value of the charge is now $7/2$, the smallest allowed representation for the bulk fermions has now spin $j_F = 7/2$.

Assuming as before $x_l \gg 1$ to simplify the analysis of the effects on order one coefficients due to wave-function corrections, the induced mass matrices $\mathcal{M}^u$ and $\mathcal{M}^d$ are given by eqs. (7.50) and (7.51) with:

$$\widetilde{Y}^d = 4\,\lambda \times \begin{pmatrix} \frac{\sqrt{7}}{4}\lambda^5 & -\frac{\sqrt{21}}{4}\lambda^4 & -\frac{\sqrt{35}}{4}\lambda^3 \\ -\frac{3}{2}\lambda^4 & \frac{5\sqrt{3}}{4}\lambda^3 & \sqrt{5}\,\lambda^2 \\ \sqrt{5}\,\lambda^2 & -\frac{\sqrt{15}}{2}\lambda & -1 \end{pmatrix}, \tag{7.91}$$

$$\widetilde{Y}^u = 4\,\lambda \times \begin{pmatrix} \frac{1}{4}\lambda^6 & \frac{\sqrt{7}}{4}\,\lambda^5 & -\frac{\sqrt{35}}{4}\,\lambda^3 \\ -\frac{\sqrt{7}}{4}\,\lambda^5 & -\frac{3}{2}\lambda^4 & \sqrt{5}\,\lambda^2 \\ \frac{\sqrt{35}}{4}\,\lambda^3 & \sqrt{5}\,\lambda^2 & -1 \end{pmatrix}, \tag{7.92}$$

The mass matrices that are obtained in this case still have the problem of a too low overall scale, but it is now possible to reproduce mass ratios and mixing angles with reasonable values of the parameters (except for the down quark mass which is too low).

7.4.2 Example with improved overall scale

The problem of the small overall scale can be solved by introducing, in addition to a pair of bulk fermions that are flavour-charged and induce general hierarchical mass matrices, an extra pair of bulk fermions that are flavour-neutral and contribute therefore only to the mass of flavour-neutral states. Assigning third-generation quarks a vanishing charge, neutral bulk fermions will only contribute to the $(3,3)$ entries of quark masses. If charged bulk fermions are heavier, all the other entries will be additionally suppressed by a factor $e^{-\pi R(M_l^C - M_l^N)}$, where M_l^C and M_l^N stand for the masses of charged and neutral bulk fermions respectively. It is clear that in this case the mass ratio between the third and the first two generations is not a prediction of the flavour model any more, but stems from the exponential factor $e^{-\pi R(M_l^C - M_l^N)}$. Taking into account this extra suppression, we can choose for example

$$Y^d \sim \begin{pmatrix} \lambda^5 & \lambda^4 & \lambda^3 \\ \lambda^4 & \lambda^3 & \lambda^2 \\ \lambda^2 & \lambda & 1 \end{pmatrix} , \qquad Y^u \sim \begin{pmatrix} \lambda^6 & \lambda^4 & \lambda^3 \\ \lambda^5 & \lambda^3 & \lambda^2 \\ \lambda^3 & \lambda & 1 \end{pmatrix} . \tag{7.93}$$

The simplest flavour charge assignment for the brane fermions that realize these is

$$q_I = \big\{-3, -2, 0\big\} , \qquad d_I = \big\{2, 1, 0\big\} , \qquad u_I = \big\{3, 1, 0\big\} . \tag{7.94}$$

The smallest allowed representation for the charged bulk fermions has in this case spin $j_F = 3$.

These charged states give a contribution to the mass matrices $\mathcal{M}^u$ and $\mathcal{M}^d$ given by eqs. (7.51) and (7.50) with

$$\widetilde{Y}^d \simeq \begin{pmatrix} -\sqrt{6}\,\lambda^5 & \sqrt{15}\,\lambda^4 & 2\sqrt{5}\,\lambda^3 \\ -5\,\lambda^4 & 2\sqrt{10}\,\lambda^3 & \sqrt{30}\,\lambda^2 \\ -2\sqrt{3}\,\lambda & \sqrt{30}\,\lambda & -1 \end{pmatrix} , \tag{7.95}$$

$$\widetilde{Y}^u = \begin{pmatrix} -\lambda^6 & \sqrt{15}\,\lambda^4 & 2\sqrt{5}\,\lambda^3 \\ -\sqrt{6}\,\lambda^5 & 2\sqrt{10}\,\lambda^3 & \sqrt{30}\,\lambda^2 \\ 2\sqrt{5}\,\lambda^3 & -2\sqrt{3}\,\lambda & -1 \end{pmatrix} . \tag{7.96}$$

For the corresponding flavour-neutral states, if we stick to the $SU(3)_W$ representations used in ref. [22], we still have a problem with the top mass, which remains too low. As an illustrative example, one can choose a rank 6 symmetric representation for the flavour-neutral fermion coupling to the top quark, even though one should check that the cutoff is not lowered too much by the presence of fermions in large representations of $SU(3)_W$. With this caveat, the situation improves, and we can reproduce all masses and mixing angles with reasonable values of the parameters, except again for the down quark which tends to be too light.

7.4.3 FCNC processes and CP violation

Since there is a mixing between brane and bulk fermions, tree-level FCNC couplings to the Z boson are expected to arise. On general grounds, they will be suppressed by α^2 and by the appropriate power of β. It remains to be seen whether in any specific model this suppression is sufficient to guarantee a successful description of FCNC phenomena: to this aim, we are presently carrying out a full one-loop phenomenological analysis.

In all the above discussions, for simplicity, we have taken the ϵ couplings to be real. In general, they are complex numbers and their phases enter the effective mass matrices and the CKM matrix. The strength of CP violation then depends on the size and phases of ϵ parameters, and can be estimated in any specific model.

7.5 Conclusion

We have proposed a mechanism to implement flavour symmetries in gauge-Higgs unification models. In five-dimensional orbifold constructions the only possibility consists in a flavour $SU(2)_F$ symmetry broken to $U(1)_F$ by the orbifold projection and then to nothing via a compactification twist. Assuming that the problems connected to electroweak symmetry breaking in gauge-Higgs unification were solved, our proposal can successfully predict the orders of magnitude of all mass ratios and mixing angles. Quantitative agreement can be obtained with reasonable values of all relevant parameters. We stress that this class of models is much more constrained than ordinary FN abelian flavour models because of the higher-dimensional non-Abelian nature of the flavour symmetry. We are presently investigating the phenomenology of FCNC processes in this kind of construction, both at the tree and the one-loop levels.

An interesting possibility would be to implement our idea in the framework of warped five-dimensional models or in six-dimensional orbifolds, in which electroweak symmetry breaking seems more successful (see for instance [139–141] and [30–33, 67]).

Conclusions

Looking for an alternative explanation of the electro-weak hierarchy problem, we have concentrated our attention on models with $D > 4$ dimensions. We have focused on the "gauge-Higgs unification" scenario, in which the 4-dimensional scalar fields are identified with some extra components of a D-dimensional gauge boson.

Five dimensional orbifold construction has been the framework to implement a flavour symmetry in the framework of gauge-Higgs unification models. Our idea consisted in considering a $SU(2)_F$ global symmetry broken to $U(1)_F$ by the orbifold projection and then to nothing via periodicity conditions. At the orbifold fixed points, the residual $U(1)_F$ mimics the standard Froggatt-Nielsen flavour abelian symmetry. Assuming that the problems connected to the electroweak symmetry breaking were solved, our proposal successfully predict the orders of magnitude of all mass ratios and mixing angles. The main weakness is the requirement of large matter representations, reducing the range of validity of the effective theory. We stress that this class of models is much more constrained than ordinary Froggatt-Nielsen abelian flavour models, because of the higher dimensional non-abelian nature of the flavour symmetry explored here.

Motivated by the difficulties in building electroweak realistic models in the context of orbifold compactifications, we have focused on an alternative: compactification in the presence of a gauge background, exploring in detail the implementation of chirality and symmetry breaking. Most of the original work included here is related to the latter problem.

Our setting has been a $U(N)$ gauge theory on a six-dimensional space-time of the type $\mathcal{M}_4 \times \mathcal{T}^2$. The choice of $U(N)$ instead of $SU(N)$ is necessary, since on a two-dimensional torus a simply-connected gauge group does not admit stable non-zero field strength configurations able to produce chirality in four dimensions. The obtention of chirality, therefore, forces us to enlarge the gauge group including a non simply-connected component: $U(1) \in U(N)$. As well as inducing chirality, a stable magnetic background associated with the abelian subgroup $U(1) \in U(N)$ affects the non-abelian subgroup $SU(N) \in U(N)$, giving rise to a non-trivial *'t Hooft non-abelian flux*.

A non-trivial 't Hooft non-abelian flux induces in general some non-trivial constraints to the $SU(N)$ Scherk-Schwarz periodicity conditions around non-contractible cycles of $\mathcal{T}^2$: the *'t Hooft consistency conditions*. The 't Hooft consistency conditions admit coordinate-dependent and constant solutions.

Whereas the trivial 't Hooft non-abelian flux case is well-known in literature, the phenomenological implications of the non-trivial case are a novel feature.

First of all, using the analogy with the harmonic oscillator, we have re-obtained the well-known result that all $SU(N)$ stable vacua compatible with four-dimensional Poincaré invariance, and zero four-dimensional instanton number, have zero energy regardless of the choice of the periodicity conditions on the torus.

We have, then, studied the classical zero-energy vacua compatible with a given set of periodicity conditions, satisfying the 't Hooft consistency conditions. To do that, we have introduced, for both the case of trivial and non-trivial 't Hooft non-abelian flux, the *background symmetric gauge*: the gauge in which the stable $SU(N)$ zero energy background configuration is trivial ($B_M = 0$) and all the physical informations are contained in the periodicity conditions. In particular, we have explicitly proved that, for $SU(N)$ on $\mathcal{T}^2$, it is always possible to work in such a gauge and that the resulting periodicity conditions are always constant.

This result has important consequences:

- The two classes of solutions of the 't Hooft consistency conditions are gauge equivalent: coordinate-dependent periodicity conditions are equivalent to constant ones. For trivial 't Hooft flux, they are equivalent to constant Scherk-Schwarz boundary conditions, associated to continuous Wilson lines. For the case of non-trivial 't Hooft flux, the coordinate-dependent boundary conditions can be traded instead by constant Scherk-Schwarz boundary conditions, associated to discrete Wilson lines.

- We have shown that to catalogue the possible vacua is completely equivalent to find all the non equivalent *constant* solutions of 't Hooft consistency conditions. Indeed, for a system with given periodicity conditions, a classical zero-energy vacuum exists for each gauge inequivalent set of constant solutions of the 't Hooft consistency conditions.

- In the background symmetric gauge, the symmetries of each stable vacuum are given by the symmetries of the constant periodicity conditions.

The number of vacua, the residual symmetries and the nature of the symmetry breaking mechanism depend on the value of the 't Hooft non-abelian flux:

- For trivial 't Hooft flux, there is a continuum of vacua, degenerate at the classical level with the $SU(N)$ symmetric one, as it is known. The symmetry breaking is rank-preserving and spontaneous. It is exactly the Hosotani mechanism in six dimensions [60–62].

- For non-trivial 't Hooft flux, there is a finite number of vacua and $SU(N)$ is broken in all of them. The symmetry breaking is rank-lowering and the 't Hooft consistency conditions forbid to interpret it as spontaneous symmetry breaking. A novel result of this thesis is the explicit proof of the symmetry breaking pattern and the four-dimensional mass spectrum.

In the case of $U(N)$ on $\mathcal{T}^2$ with a $U(1) \subset U(N)$ magnetic background, we were able to determine the vacua and the residual symmetries using theoretical arguments. The same theoretical arguments do not hold for a general non-simply connected group. For example, there is no reason a priori for describing all vacua in terms of some constant periodicity conditions. In these cases, an effective field-theory treatment of a system subject to coordinate-dependent boundary conditions will be necessary in order to determine stable vacua and residual symmetries.

We have treated explicitly the case of $SU(2)$ on a torus with a background compatible with coordinate-dependent boundary conditions. A field theory analysis has allowed us, in addition, to explicitly solve the Nielsen-Olesen instability on a two dimensional torus.

For the obtention of the four-dimensional effective Lagrangian, all couplings have been taken into account, including *all* quartic and cubic terms mixing Kaluza-Klein modes and Landau levels. Those terms are shown to be essential in the determination of the stable minimum of the potential and its symmetries. The corresponding integrals over the extra-dimensional space have been obtained analytically for all modes, for the first time. Furthermore, we have defined gauge-fixing Lagrangians, appropriate when both Kaluza-Klein modes and Landau levels are simultaneously present. The computations have been performed in different gauges and the issue of gauge fixing has been clarified in depth. These technical tools will be necessary when groups other than $SU(N)$ will be considered.

The system is seen to evolve dynamically from the unstable background configuration towards a stable and non-trivial background of zero energy. This happens through an infinite chain of vacuum expectation values of the four-dimensional scalar fields. The resulting spectra do show explicitly the symmetries expected from the theoretical analysis mentioned above, for the case of $SU(N)$ with constant boundary conditions, supporting the strength of our effective theory analysis.

We have also analyzed the problem of symmetry breaking at the quantum level. In particular, we have explicitly computed the one loop effective potential using the Heat Kernel teqnique. This type of computation takes place in coordinate space and results in a very useful instrument to distinguish contributions coming from local and non-local diagrams. Recent literature [113,114] has evidenced that, at 1-loop, the extra-dimensional and the four-dimensional computation of the same quantity do not necessarily coincide. In particular the counterterms necessary to remove the 1-loop divergences show some differences in the two cases. Such differences remain when all Kaluza-Klein modes are included the four-dimensional computation. For this reason, in our computation we have adopted both the extra- and the four-dimensional point of view. It has then been possible to evidence that the non-local and finite effects can be equivalently described using both points of view.

For both trivial and no-trivial 't Hooft flux, cases, the symmetry breaking mechanism can be interpreted in terms of periodicity conditions exclusively. Quantum corrections due to diagrams that do not wind at least once around some non-contractible loops of $\mathcal{T}^2$ are insensitive to the periodicity conditions: they are $SU(N)$ symmetric and sensitive to the microscopic dynamics. All quantum corrections containing $SU(N)$ symmetry breaking parameters are associated to diagram wrapping around some non-contractible cycle of $\mathcal{T}^2$ and are expressed in terms of $SU(N)$ symmetric non-local operators: the trace of powers of Wilson loops. This type of quantum corrections are calculable and naturally cut-offed by the inverse of the cycle length. Because of their non-local character, they are insensitive to the microscopic (high energy) dynamics. This analysis of the quantum stability for the case of a non-trivial 't Hooft flux, is also a novel aspect of this thesis.

Flux compactification seems to be a promising framework for physics beyond the Standard Model. It allows, indeed, to obtain chirality as well as to implement a rank-lowering symmetry breaking which is not affected by the hierarchy problem. However, the obtention of a realistic model still remains a non-trivial issue. In our analysis, we have run into some phenomenological drawbacks that need to be investigated in more detail.

For instance, this symmetry breaking mechanism does not distinguish between ordinary and extra components of a higher-dimensional gauge boson. It then turns out that for each four-dimensional gauge boson there exists a scalar partner degenerate in mass. This result holds for both trivial and non-trivial 't Hooft fluxes.

In the case of non-trivial 't Hooft flux, although the symmetry breaking is rank-lowering and ultraviolet-insensitive, to find a realistic pattern of electroweak symmetry breaking remains a non-trivial issue. In order to reproduce the Standard model, indeed, we cannot start in the extra dimensions directly with the electroweak gauge group $SU(2) \times U(1)$: the non-trivial 't Hooft flux symmetry breaking patterns cannot reproduce the electroweak symmetry breaking $SU(2) \times U(1) \to U(1)_{em}$. We have to consider, therefore, an extra dimensional gauge group $U(N)$ large enough to include $SU(2) \times U(1)$. On the other hand, in such type of construction, all dimensional quantities are a function of the value of the two radii R_1, R_2 of the torus and of N and therefore they are all of the same order of magnitude. This result implies that all massive gauge bosons ($W^{\pm}$, Z_0 and extra-SM gauge bosons) should be degenerate.

All these aspects could find a common solution enlarging the minimal scenario considered in this thesis, including for instance new different extra dimensional stable configurations: this can be realized, for example, changing either the initial gauge group or the characteristics (topology, number of dimensions) of the compactified manifold.

Another interesting possibility to explore is the putative presence of fermions in the bulk. Such bulk fermions would interact with the $U(1) \in U(N)$ magnetic background and, consequently, their wave functions would be "approximately-localized" on the extra dimensions. Here, "approximately" means a gaussian wave function with a width proportional to $1/\mathcal{A}$, where $\mathcal{A}$ is the area of the torus. At the same time, these "approximately-localized" fermions interact with the $SU(N) \in U(N)$ gauge bosons inducing "approximately-localized" four-dimensional gauge and scalar operators. The latter could have interesting phenomenological consequences and is under study.

Conclusiones

El problema de la Jerarquía que caracteriza la ruptura de simetría electrodébil ha sido la principal motivación para el estudio de posibles extensiones del Modelo Estándar en el contexto de las dimensiones extra $D > 4$. En particular, el escenario elegido para este trabajo ha sido la "unificación gauge-Higgs". En este en tipo de construcción, los escalares cuadridimensionales derivan de las componentes extra de un bosón gauge que vive en un espacio-tiempo D-dimensional.

Las teorías con cinco dimensiones donde la quinta está compactificada en un orbifoldio, han sido el escenario de parte de nuestro trabajo. En particular, nos hemos ocupados de la implementación de una simetría de sabor en el contexto de la unificación gauge-Higgs. Más detallatamente, nuestra idea consiste en considerar una simetría global $SU(2)_F$ que se rompe a $U(1)_F$ mediante la proyección de orbifoldio y, seguidamente, al grupo trivial gracias a condiciones de periodicidad no-triviales. En los puntos fijos del orbifoldio, la simetría residual $U(1)_F$ se comporta como las tradicionales simetrías abelianas de sabor *á la* Froggatt y Nielsen. Suponiendo que los problemas relacionados con la ruptura electrodébil hayan sido solucionados, nuestra propuesta predice satisfactoriamente los órdenes de magnitud de todos los cocientes de las masas y de todos los ángulos de mezcla. La principal debilidad de este tipo de construcción es la necesidad de representaciones grandes para los campos de materia, reduciendo de esta manera el intervalo de validez de la teoría efectiva.

Debido a las dificultades al reproducir la ruptura electrodébil en el contexto de la compactificación mediante orbifoldio, hemos focalizado nuestra atención sobre diferentes compactificaciones. En concreto, hemos estudiado la compactificación de dimensiones extra sobre variedades no-simplemente conexas donde viven campos de fondo. En este contexto hemos analizado los problemas de la quiralidad y de la ruptura de simetría. La mayor parte del trabajo original incluido en esta tesis está relacionado con este último punto.

Nuestro punto de partida ha sido una teoría gauge $U(N)$ en un espacio de seis dimensiones del tipo $\mathcal{M}_4 \times \mathcal{T}^2$. La elección del grupo $U(N)$ es una consecuencia del hecho de que en un toro de dos dimensiones un grupo simplemente conexo (por ejemplo

$SU(N)$) no admite configuraciones estables con *field strength* diferente de cero, necesarias para la obtención de la quiralidad en cuatro dimensiones. La quiralidad en cuatro dimensiones obliga a agrandar el grupo gauge incluyendo una componente no-simplemente conexa: $U(1) \in U(N)$. Además de producir quiralidad, la presencia de un campo de fondo magnético estable asociado con el subgrupo abeliano $U(1) \in U(N)$ influye también sobre el subgrupo no-abeliano $SU(N) \in U(N)$, dando lugar a un *flujo no-abeliano de 't Hooft* no-trivial.

Un flujo de 't Hooft no-trivial implica restricciones no-triviales para las posibles condiciones de periodicidad de Scherk-Schwarz alrededor de los ciclos no-contráctiles de $\mathcal{T}^2$, llamadas las *condiciones de consistencia de 't Hooft*. Estas condiciones admiten dos clases de soluciones: constantes y dependientes de las coordenadas. El principal objetivo de esta tesis ha sido entender cómo la presencia de condiciones de periodicidad (solución de las condiciones de consistencia de 't Hooft) afecta a la teoría y en particular a la energía del vacío, al número y tipo de vacíos posibles, a las simetrías residuales o a la estabilidad cuántica de la ruptura de simetría.

El caso de flujo de 't Hooft trivial ha sido analizado detallatamente en la literatura. El análisis de las implicaciones fenomenológigas del caso no-trivial es uno de los aspectos novedosos de esta tesis.

Usando la analogía con el oscilador armónico, hemos obtenido primeramente el conocido resultado de que todos los vacíos estables de $SU(N)$ compatibles con la invariancia de Poincaré cuadridimensional[7] tienen energía cero independentemente de la elección de las condiciones de periodicidad.

El paso siguiente ha sido analizar los vacíos clásicos de energía cero compatibles con condiciones de periodicidad dadas, que son soluciones de las condiciones de consistencia de 't Hooft. Para simplificar este estudio hemos introducido para ambos los casos de flujo de 't Hooft trivial y no-trivial, el *gauge de fondo simétrico*: el gauge donde las configuraciones de fondo con energía cero son triviales ($B_M = 0$) y toda la información física se encuentra en las condiciones de periodicidad. Hemos demostrado explícitamente que en el caso de $SU(N)$ sobre $\mathcal{T}^2$ es siempre posible trabajar en este gauge y que las condiciones de periodicidad resultan ser siempre constantes.

Este resultado tiene importantes consecuencias:

- Las dos clases de soluciones de las condiciones de consistencia de 't Hooft son equivalentes: las condiciones de periodicidad que dependen de las coordenadas son equivalentes a las constantes. En el caso de flujo de 't Hooft trivial, las condiciones de periodicidad que dependen de las coordenadas son equivalentes a condiciones de periodicidad constantes asociadas a líneas de Wilson continuas. En el caso de flujo

[7]Estamos considerando implícitamente configuraciones con número de instantones cuadridimensionales igual a cero

de 't Hooft no-trivial, las condiciones de periodicidad que dependen de las coordenadas pueden ser entendidas como condiciones de periodicidad constantes asociadas a líneas de Wilson discretas.

- Catalogar los posibles vacíos es completamente equivalente a catalogar todas las soluciones *constantes* de las condiciones de consistencia de 't Hooft. Para un sistema con determinadas condiciones de periodicidad existe un vacío clásico de energía cero para cada conjunto no equivalente de soluciones de las condiciones de consistencia de 't Hooft.

- En el gauge de fondo simétrico, las simetrías de cada vacío estable coresponden a las simetrías de las condiciones de periodicidad constantes.

El número de vacíos, las simetrías residuales y la naturaleza de la ruptura de simetría dependen del valor del flujo de 't Hooft:

- En el caso de flujo de 't Hooft trivial, existe un continuo de vacíos degenerados a nivel clásico con el vacío $SU(N)$ simétrico. La ruptura de simetría es espontánea y preserva el rango del grupo. Éste es perfectamente el mecanismo de Hosotani en seis dimensiones.

- En el caso de flujo de t' Hooft no-trivial, existe un numero finito de vacíos. Todos estos vacíos implican cierto grado de ruptura del grupo inicial $SU(N)$. La ruptura de simetría disminuye el rango del grupo y las condiciones de consistencia de 't Hooft impiden interpretar la misma ruptura de simetría como una ruptura espontánea. Uno de lo resultados novedosos de esta tesis es la demonstación explícita del patrón (*pattern*) de ruptura de simetría y del espectro de masa en cuatro dimensiones.

En ambos casos, el mecanismo de ruptura de simetría puede ser interpretado exclusivamente en términos de las condiciones de periodicidad. Las correcciones cuánticas, debidas a diagramas que no se arrollan al menos una vez alrededor de un ciclo no-contráctil de $\mathcal{T}2$, presentan las siguientes características:

- Resultan insensibles a las condiciones de periodicidad.

- Respetan toda la simetria $SU(N)$.

- Son sensibles a la dinámica microscópica.

Todas las correcciones cuánticas que contienen parámetros que rompen $SU(N)$ están asociadas a diagramas que se arrollan alrededor de ciclos no-contráctiles de $\mathcal{T}^2$ y están expresadas en términos de operadores no-locales, esto es, la traza de potencias del *loop* de Wilson. Este tipo de correcciones cuánticas resulta naturalmente acotadas por la inversa de la longitud del ciclo. Los parámetros de ruptura de simetría que aparecen en las condiciones de periodicidad son insensibles a la dinámica microscópica (de altas energías).

En el caso de $U(N)$ sobre $\mathcal{T}^2$ en presencia de un campo de fondo magnético asociado con el subgrupo $U(1) \subset U(N)$, es posible determinar los posible vacíos y las simetrías residuales utilizando argumentos teóricos.

Estos mismos argumentos teóricos no resultan necesariamente válidos para un grupo general no-simplemente conexo. Por ejemplo, a priori no existe alguna razón que garantice la posibilidad de describir todos los vacíos de la teoría en términos de condiciones de periodicidad constantes. En estos casos, será necesario desarollar la teoría efectiva para un sistema sujeto a condiciones de periodicidad que dependen de las coordenadas, con el objeto de determinar los vacíos estables y las simetrías residuales.

Hemos realizado este análisis, de forma explícita, para el caso de una teoría gauge $SU(2)$ sobre un toro de dos dimensiones en el que vive un campo de fondo compatible con condiciones de periodicidad dependientes de las coordenadas.

El análisis de la teoría efectiva nos ha permitido resolver explícitamente la inestabilidad de Nielsen-Olesen en un toro de dos dimensiones.

Para obtener el lagrangiano efectivo en cuatro dimensiones, hemos considerado *todos* los términos cúbicos y cuárticos que mezclan modos de Kaluza-Klein y niveles de Landau. Estos términos resultan imprescindibles para determinar el mínimo estable del potencial y sus simetrías. Las integrales sobre las dimensiones extra han sido obtenidas de manera analítica para todos los modos. Además, hemos definido términos de *gauge fixing* compatibles con la presencia simultánea de niveles de Landau y modos de Kaluza-Klein.

El cálculo ha sido realizado para varias elecciones del gauge y el problema del *gauge fixing* ha sido discutido en detalle. Estas herramientas matemáticas serán necesarias para futuros estudios de grupos gauge diferentes de $SU(N)$.

En el análisis fenomenológico es posible comprobar cómo el sistema evoluciona dinámicamente desde una configuración de fondo inestable hacia una configuración estable, no-trivial y con energía cero. Se alcanza un nuevo vacío estable a través de una cadena infinita de valores esperados en el vacío (*vev's*) para los campos escalares cuadridimensionales. El espectro resultante muestra explícitamente las simetrías predichas al utilizar los argumentos teóricos en el caso de $SU(N)$. Este acuerdo respalda la solidez de la teoría efectiva.

La compactificación en presencia de flujo magnético parece ser un escenario prometedor para acomodar nueva física más allá del Modelo Estandard. En este escenario es posible obtener quiralidad y al mismo tiempo implementar un mecanismo de ruptura de simetría que no esté afectado por el problema de la Jerarquía. Sin embargo, la obtención de modelos realistas persiste como un problema no-trivial. Nuestro análisis ha evidenciado algunas complicaciones que necesitan ser investigadas con más detalle.

Por ejemplo, la ruptura de simetría en presencia de campo de fondo magnético no distingue entre las componentes ordinarias y extra de un bosón de gauge extradimensional. Como consecuencia, para cada bosón de gauge cuadridimensional existe un compañero escalar con la misma masa. Este resultado es común para los casos de flujo trivial y no-trivial de 't Hooft.

En el caso de flujo de 't Hooft no-trivial, reproducir la ruptura electrodébil resulta bastante complicado a pesar de que la ruptura de simetría sea capaz de disminuir el rango del grupo y resulte insensible a la física ultravioleta. No es posile, por ejemplo, comenzar desde el primer momento con el grupo de simetría del Modelo Estándar en dimensiones extra puesto que la ruptura de simetría inducida por el flujo de 't Hooft no-trivial impide la ruptura $SU(2) \times U(1) \rightarrow U(1)_{em}$. Tenemos que comenzar por tanto con un grupo $U(N)$ lo suficientemente grande como para incluir $SU(2) \times U(1)$.

Por otro lado, en este tipo de construcción existe una sola escala de energía. Esta escala depende del área del toro y del valor del flujo de 't Hooft. Como consecuencia, todas las cantidades dimensionales que rompen la simetría dependen de una única escala y resultan en su totalidad del mismo orden de magnitud. Este resultado implica que las masas de los bosones gauge ($W^{\pm}$, Z_0 y bosones gauge *extra-Standard-Model*) están degeneradas.

Todos estos aspectos podrían encontrar una única solución extendiendo el escenario mínimo considerado en esta tesis; incluyendo, por ejemplo, nuevas configuraciones extradimensionales que sean estables. Se puede implementar esta idea modificando el grupo gauge inicial o las caracteristícas (topología, número de dimensiones...) de la variedad compactificada.

Appendix A

Landau Levels

In this appendix we derive the wave functions for the Landau levels on a $2D$ torus [77,142], with charge $q > 0$, defined as the solutions of the eigenvalue problem

$$a_+^\dagger a_+ \, f^{+\,(j)}(y) \quad = \quad j\, f^{+\,(j)}(y)\,, \tag{A.1}$$

where $a_+^\dagger$ and a_+ are given in eq.(3.35). They obey the boundary conditions

$$\begin{aligned} f^{+\,(j)}(y\,+\,l_1) \quad &= \quad e^{i\pi d \frac{y_2}{l_2}}\, f^{+\,(j)}(y)\,, &\text{(A.2)}\\ f^{+\,(j)}(y\,+\,l_2) \quad &= \quad e^{-i\pi d \frac{y_1}{l_1}}\, f^{+\,(j)}(y)\,, &\text{(A.3)}\end{aligned}$$

where $d = q\left(k + \frac{m}{N}\right)$. It is easy to compute first the zero mode, satisfying

$$a_+ \, f^{+(j=0)}(y) = 0 \tag{A.4}$$

and, subsequently, obtain all the heavier solutions by recursively applying the creation operator $a^\dagger$:

$$f^{+\,(j+1)}(y) \quad = \quad \sqrt{\frac{1}{j+1}}\, a_+^\dagger \, f^{+\,(j)}(y)\,. \tag{A.5}$$

A possible ansatz for the wave function $f^{+\,(j=0)}(y)$, compatible with the periodicity condition along the direction y_1 in Eq..(A.2), is

$$f^{+\,(j=0)}(y) = \sum_{n=-\infty}^{\infty} c_n(y_2) e^{i\pi d \frac{y_1 y_2}{l_1 l_2}} e^{2\pi i n \frac{y_1}{l_1}}\ . \tag{A.6}$$

The periodicity condition along the direction y_2, Eq..(A.3), implies that d *must be an* *integer* and the coefficients $c_n(y_2)$ must satisfy the periodicity condition:

$$c_n(y_2 + l_2) = c_{n+d}(y_2)\ . \tag{A.7}$$

The coefficients $c_n(y_2)$ are explicitly obtained after the substitution of Eq..(A.6) into Eq..(A.4), giving

$$\partial_2 c_n(y_2) = \left(-\frac{2\pi\, d}{l_1 l_2} y_2 - \frac{2\pi n}{l_1} \right) c_n(y_2)\,, \tag{A.8}$$

with solution

$$c_n(y_2) = A_n e^{-\frac{\pi\, d}{l_1 l_2} y_2^2 - \frac{2\pi n}{l_1} y_2}\ . \tag{A.9}$$

The coefficient A_n is determined by the periodicity condition for the $c_n(y_2)$, Eq..(A.7), implying

$$A_{n+d} = A_n e^{-\pi \frac{l_2}{l_1}(2n+d)}\,, \tag{A.10}$$

whose solution is

$$A_n = b_n e^{-\pi \frac{l_2}{l_1} \frac{n^2}{d}}\,, \tag{A.11}$$

where the constants b_n satisfy $b_{n+d} = b_n$. It exists, therefore, d arbitrary constant coefficients and, consequently, d independent solutions for the zero mode. We will characterize them by the integer number ρ, $\rho = 0, ..., d-1$, as described in Sect. 3.

All in all, the lightest wave function can be written as

$$f^{+\,(j=0)}(y) \ \ = \ \ \sum_{\rho=0}^{d-1} b_\rho\, f^{+\,(j=0,\rho)}(y)\,, \tag{A.12}$$

where b_ρ are arbitrary coefficients subject to the normalization condition

$$\sum_{\rho=0}^{d-1} |b_\rho|^2 = 1\,, \tag{A.13}$$

and the functions $f^{+\,(j=0,\rho)}(y)$ are given by

$$f^{+\,(j=0,\rho)}(y) = \left(\frac{2d}{l_1^3\, l_2} \right)^{\frac{1}{4}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi d}{l_1 l_2}(y_2 + n l_2 + \frac{\rho l_2}{d})^2} e^{2\pi i (dn+\rho)\frac{y_1}{l_1}} e^{i \frac{\pi d}{l_1 l_2} y_1 y_2}\ . \tag{A.14}$$

Notice that for $d > 1$ the different independent solutions $f^{+\,(j,\rho)}(y)$ are localized at different points of the extra dimensions.

Finally, the expression of the heavier modes resulting from Eq..(A.5) reads:

$$f^{+\,(j,\rho)}(y) = \left(\frac{2d}{l_1^3\, l_2}\right)^{\frac{1}{4}} \frac{(-i)^j}{\sqrt{2^j\, j!}}\, e^{i\frac{\pi d}{l_1 l_2} y_1 y_2}$$
$$\cdot \sum_{n=-\infty}^{\infty} e^{-\frac{\pi d}{l_1 l_2}(y_2 + n l_2 + \frac{\rho l_2}{d})^2}\, e^{2\pi i \frac{y_1}{l_1}(dn+\rho)}\, H_{j,\rho}\left[\sqrt{\frac{2\pi d}{l_1 l_2}}\left(y_2 + n l_2 + \frac{\rho l_2}{d}\right)\right], \tag{A.15}$$

with $H_{j,\rho}(y)$ being the Hermite polynomials.

Appendix B

Integrals

We summarize the integrals of the extra dimensional wave functions, necessary to explicitly obtain the effective coefficients of the 4D theory.

- Two-field integrals:

$$\int_{T^2} f^{3\,(n_1,n_2)}\, f^{3\,(m_1,m_2)} d^2y \quad = \quad \delta_{n_1,-m_1}\,\delta_{n_2,-m_2}\,, \tag{B.1}$$

$$\int_{T^2} f^{+\,(j_1,\rho_1)}\, f^{-\,(j_2,\rho_2)} d^2y \quad = \quad \delta_{j_1,j_2}\,\delta_{\rho_1,\rho_2}\,, \tag{B.2}$$

  where $f^{3\,(n_1,n_2)}$ and $f^{+\,(j,\rho)}$ are respectively given by eq. (3.31) and eq. (3.39).

- Three-field integrals:
  if $\frac{\rho_2-\rho_1-n_1}{d} \notin \mathbb{Z}$,

$$I^{(3)}[j_1,\rho_1,j_2,\rho_2,n_1,n_2] = \int_{T^2} f^{+\,(j_1,\rho_1)}\, f^{-\,(j_2,\rho_2)}\, f^{3\,(n_1,n_2)}\, d^2y = 0\,, \tag{B.3}$$

  else

$$\begin{aligned} I^{(3)}[j_1,\rho_1,j_2,\rho_2,n_1,n_2] \quad &= \quad l_1\sqrt{\frac{R}{\mathcal{A}^2}}\, e^{-2\pi i\frac{\rho_1 n_2}{d}}\, e^{-\pi i\frac{n_1 n_2}{d}}\, e^{-\frac{\pi}{2d}(\frac{n_2^2}{R}+Rn_1^2)}\, \frac{\sqrt{j_1!j_2!}}{2^{j_1+j_2}} \qquad \text{(B.4)} \\ &\times \sum_{k=0}^{j_2}\sum_{k_1=0}^{[\frac{j_1}{2}]}\sum_{k_2=0}^{Min[k,j_1-2k_1]} \frac{2^{k_2}(-1)^{k_1} i^{j_1+k-2k_1-2k_2}}{k_1!k_2!(j_2-k)!(j_1-2k_1-k_2)!(k-k_2)!} \\ &\times H_{j_1+k-2k_1-2k_2}\left[\sqrt{\frac{\pi}{d}}\left(\frac{n_2}{\sqrt{R}}+i\sqrt{R}n_1\right)\right]\, H_{j_2-k}\left[2\sqrt{\frac{\pi R}{d}}n_1\right]\,, \end{aligned}$$

  where $\mathcal{A} = l_1 l_2$ and $R = l_2/l_1$.

- Four-field integrals with two charged and two neutral fields:

$$I_{NC}^{(4)}[j_1,\rho_1,j_2,\rho_2,n_1,n_2,m_1,m_2] \equiv \int_{T^2} f^{+(j_1,\rho_1)}\, f^{-(j_2,\rho_2)}\, f^{3\,(n_1\,n_2)}\, f^{3\,(m_1\,m_2)}\, d^2y = I^{(3)}[j_1,\rho_1,j_2,\rho_2,n_1+m_1,n_2+m_2]\,. \quad \text{(B.5)}$$

- Four-field integrals with four charged fields:
  when $\frac{\rho_1+\rho_3-\rho_2-\rho_4}{d} \notin \mathbb{Z}$,

$$I_C^{(4)}[j_1,\rho_1,j_2,\rho_2,j_3,\rho_3,j_4,\rho_4] \equiv \int_{T^2} f^{+(j_1,\rho_1)}\, f^{-(j_2,\rho_2)}\, f^{+(j_3,\rho_3)}\, f^{-(j_4,\rho_4)}\, d^2y = 0\,, \quad \text{(B.6)}$$

  else

$$I_C^{(4)}[j_1,\rho_1,j_2,\rho_2,j_3,\rho_3,j_4,\rho_4] = \frac{\sqrt{dR}}{\mathcal{A}}\,\frac{\sqrt{j_1!j_2!j_3!j_4!}}{2^{j_1+j_2+j_3+j_4}} \sum_{p,k=-\infty}^{\infty} e^{-\pi dR\left[(\frac{\rho_1-\rho_2}{d}-k)^2+(\frac{\rho_1-\rho_4}{d}-p)^2\right]}$$
$$\times \sum_{k_1=0}^{j_1}\sum_{k_2=0}^{j_2}\sum_{k_3=0}^{j_3}\sum_{k_4=0}^{j_4}\sum_{z_1=0}^{Min[k_1,k_2]}\sum_{z_2=0}^{Min[k_3,k_4]} \frac{2^{z_2-z_1+k_1+k_2}(k_1+k_2-2z_1)!\,\delta^{k_1+k_2-2z_1}_{k_3+k_4-2z_2}}{z_1!z_2!(j_1-k_1)!(j_2-k_2)!(j_3-k_3)!(j_4-k_4)!}$$
$$\times \frac{H_{j_1-k_1}\left[-\sqrt{\pi dR}(k+p+\frac{\rho_4+\rho_2-2\rho_1}{d})\right] H_{j_2-k_2}\left[-\sqrt{\pi dR}(-k+p+\frac{\rho_4-\rho_2}{d})\right]}{(k_1-z_1)!(k_2-z_1)!(k_3-z_2)!(_4-z_2)!}$$
$$\times H_{j_3-k_3}\left[\sqrt{\pi dR}(k+p+\frac{\rho_4+\rho_2-2\rho_1}{d})\right] H_{j_4-k_4}\left[\sqrt{\pi dR}(-k+p+\frac{\rho_4-\rho_2}{d})\right]\,. \quad \text{(B.7)}$$

The integrals above are related by the following completeness relationships, which we have checked numerically up to a precision better than 10^{-6}.

$$I_C^{(4)}[j_1,\rho_1,j_2,\rho_2,j_3,\rho_3,j_4,\rho_4] = \sum_{n_1,n_2=-\infty}^{\infty} I^{(3)}[j_1,\rho_1,j_2,\rho_2,n_1,n_2]\, I^{(3)}[j_3,\rho_3,j_4,\rho_4,-n_1,-n_2]\,, \quad \text{(B.8)}$$

$$I_{NC}^{(4)}[j_1,\rho_1,j_2,\rho_2,n_1,n_2,m_1,m_2] = \sum_{\rho=0}^{d-1}\sum_{j=0}^{\infty} I^{(3)}[j_1,\rho_1,j,\rho,n_1,n_2]\, I^{(3)}[j,\rho,j_2,\rho_2,m_1,m_2]\,. \quad \text{(B.9)}$$

Appendix C

$m \neq 0$ symmetry breaking pattern

In this appendix, we prove the formulae in eq. (4.61)-(4.62) regarding the possible symmetry breaking patterns that can be achieved in the case of non-trivial 't Hooft non-abelian flux m.

First of all, notice that the set of solutions of the condition in eq. (4.56) is given by the set of the 2×2 matrices

$$M = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}, \tag{C.1}$$

with integer entries $\alpha_1,\, \alpha_2,\, \beta_1,\, \beta_2 \in [-N+1, N-1]$ and having Det $M = m$.

The quantities m and $\mathcal{K} = g.c.d.(\alpha_1,\, \alpha_2,\, \beta_1,\, \beta_2)$ are invariant under the following biunimodular transformations

$$M \to U\, M\, V\,. \tag{C.2}$$

U and V can be parametrized in terms of integer numbers $n_1, n_2, ..$ and $m_1, m_2, ..$ and the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad\qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{C.3}$$

as follows

$$\begin{aligned} U &= S\, T^{n_1}\, S\, T^{n_2} \ldots \\ V &= S\, T^{m_1}\, S\, T^{m_2} \ldots \;. \end{aligned} \tag{C.4}$$

The invariance of $\mathcal{K}$ under the transformations in eq. (C.2) can be proved using the properties of the great common divisor:

$$\begin{aligned} g.c.d.(\alpha, \beta, \gamma) &= g.c.d.(\, g.c.d.(\alpha, \beta), \gamma) = g.c.d.(g.c.d.(\alpha, \gamma), \beta) \\ &= g.c.d.(g.c.d.(\beta, \gamma), \alpha) \\ g.c.d.(\alpha + n\beta, \beta) &= g.c.d.(\alpha, \beta)\,, \end{aligned} \tag{C.5}$$

where α, β, γ, n are integers.

The matrices in eq. (C.1) can be partitioned in equivalence classes, characterized by the two integer parameters m and $\mathcal{K} = g.c.d(\alpha_1, \alpha_2, \beta_1, \beta_2)$. Indeed, using the Bezout theorem, it is possible to check that any pair of matrices in eq. (C.1) with given m and $\mathcal{K}$ and parameters α_1, α_2, β_1, β_2 and α_1', α_2', β_1', β_2' respectively, can be always related by the transformations in eq. (C.2). In particular, it is always possible to convert a general matrix of the form in eq. (C.1) (characterized by m and $\mathcal{K}$) in the following way[1]

$$M = \begin{pmatrix} \frac{m}{\mathcal{K}} & 0 \\ 0 & \mathcal{K} \end{pmatrix} . \tag{C.6}$$

For eq. (C.6), the condition in eq. (4.59) that selects which gauge bosons admit zero mode (and consequently which are the residual symmetries in the case of non-trivial 't Hooft non-abelian flux) reduces to

$$\frac{m\Delta}{N\mathcal{K}} \in \mathcal{Z} \qquad \Longrightarrow \qquad \Delta = \frac{N\mathcal{K}}{m}\, i \tag{C.7}$$

$$\frac{\mathcal{K}k_\Delta}{N} \in \mathcal{Z} \qquad \Longrightarrow \qquad k_\Delta = \frac{N}{\mathcal{K}}\, j \ , \tag{C.8}$$

where i, j are integers. Remembering that Δ and k_Δ are defined modulo N, it is possible to check that eq. (C.7) admits $\mathcal{K}_1 = g.c.d.(m, N)$ independent solutions given by

$$\Delta = \frac{N}{\mathcal{K}_1}\frac{\mathcal{K}}{\mathcal{K}_2}\, i \qquad\qquad \text{for} \ \ i = 0, .., \mathcal{K}_1 - 1\,, \tag{C.9}$$

where $\mathcal{K}_2 = g.c.d.(\mathcal{K}, N) = g.c.d.(\alpha_1, \alpha_2, \beta_1, \beta_2, N)$. In particular, notice[2] that if $\mathcal{K}_2 > 1$ then $\mathcal{K} = \mathcal{K}_2$ and if $\mathcal{K}_2 = 1$ then N and $\mathcal{K}$ are coprime. Eq. (C.8), instead, has $\mathcal{K}_2$ independent solutions given by

$$k_\Delta = \frac{N}{\mathcal{K}_2}\, j \qquad\qquad \text{for} \ \ j = 0, .., \mathcal{K}_2 - 1\,. \tag{C.10}$$

[1]Notice that $m/\mathcal{K}$ is a multiple of $\mathcal{K}$, as it can be proved using the definition of $\mathcal{K}$:

$$\frac{m}{\mathcal{K}}\frac{1}{\mathcal{K}} = \frac{\alpha_1}{\mathcal{K}}\frac{\beta_2}{\mathcal{K}} - \frac{\alpha_2}{\mathcal{K}}\frac{\beta_1}{\mathcal{K}} \in \mathcal{Z}$$

[2]The quantity m can be rewritten as $m = z_m \mathcal{K}^{n_m}$ with the integer $n_m \geq 2$, z_m integer and $g.c.d.(z_m, \mathcal{K}) = 1$. Therefore if $\mathcal{K}_1 = g.c.d.(m, N) = g.c.d.(z_m \mathcal{K}^{n_m}, N) > 1$, it is possible to have two different cases:

$$\text{if} \ \ \frac{\mathcal{K}_1}{\mathcal{K}} \in \mathbf{Z}\,, \qquad \mathcal{K}_2 = g.c.d.(\mathcal{K}, N) = \mathcal{K}\,, \qquad \text{else} \qquad \mathcal{K}_2 = g.c.d.(\mathcal{K}, N) = 1$$

Since Δ and k_Δ cannot be simultaneously zero, the number of $\tau\,(\Delta, k_\Delta)$ that fulfill the conditions in eqs.(C.7)-(C.8), and then the dimension of the residual symmetry group is $\mathcal{K}_1\,\mathcal{K}_2\,-\,1$.

Now, we want to determinate the algebra associated to the residual symmetry group. To do that, we resume the algebra of our basis $\tau\,(\Delta,\,k_\Delta)$:

$$[\tau\,(\Delta,\,k_\Delta)\,,\,\tau\,(\Delta',\,k'_\Delta)] \,=\, \left(e^{\frac{2\pi i}{N}\,\Delta\,k'_\Delta}\,-\,e^{\frac{2\pi i}{N}\,\Delta'\,k_\Delta}\right)\,\tau\,(\Delta\,+\,\Delta',\,k_\Delta+k'_\Delta)\;. \tag{C.11}$$

This commutation rule, for those $\tau(\Delta, k_\Delta)$ that satisfy eqs.(C.7)-(C.8), reads

$$\begin{aligned}&\left[\tau\left(\frac{N}{\mathcal{K}_1}\frac{\mathcal{K}}{\mathcal{K}_2}\,i_1,\,\frac{N}{\mathcal{K}_2}\,j_1\right),\,\tau\left(\frac{N}{\mathcal{K}_1}\frac{\mathcal{K}}{\mathcal{K}_2}\,i_2,\,\frac{N}{\mathcal{K}_2}\,j_2\right)\right]\\ &\quad=\,\left(e^{\frac{2\pi i N}{\mathcal{K}_1\mathcal{K}_2}\,\frac{\mathcal{K}}{\mathcal{K}_2}\,i_1\,j_2}\,-\,e^{\frac{2\pi i N}{\mathcal{K}_1\mathcal{K}_2}\,\frac{\mathcal{K}}{\mathcal{K}_2}\,i_2\,j_1}\right)\,\tau\left(\frac{N}{\mathcal{K}_1}\,\frac{\mathcal{K}}{\mathcal{K}_2}\,(i_1+i_2),\,\frac{N}{\mathcal{K}_2}\,(j_1+j_2),\,\right)\;.\end{aligned} \tag{C.12}$$

As a first step, we want to individuate the dimension of the abelian subspace of this algebra, that is, the number of residual $\tau(\Delta, k_\Delta)$ that commute with all the other residual ones. To do that, we have to distinguish two different cases:

- $\frac{N}{m}\in\mathbf{Z}$: in this case necessarily $\mathcal{K}_1 = m$ and $\mathcal{K}_2 = 1$ and the commutator in eq. (C.12) is always zero, independently of the indices $i_1,\,j_1,\,i_2,\,j_2$. All $\mathcal{K}_1\,\mathcal{K}_2\,-\,1\,=\,m\,-\,1$ generators commute among them.

- $\frac{N}{m}\notin\mathbf{Z}$: the commutator in eq. (C.12) is zero for any values of $i_2,\,j_2$, if

  $$i_1\,=\,\mathcal{K}_2\,i'\;,\qquad\qquad j_1\,=\,0\,,$$

  with $i' = 1, .., \mathcal{K}_1/\mathcal{K}_2$. The algebra of the residual symmetry group has an abelian subset of dimensions $\mathcal{K}_1/\mathcal{K}_2 - 1$.

Summarizing, the condition in eq. (4.59) for the case in eq. (C.6) produces the following symmetry breaking pattern

$$SU(N)\quad\rightarrow\quad\mathcal{G}\,\times\,U(1)^{\frac{\mathcal{K}_1}{\mathcal{K}_2}-1}\,, \tag{C.13}$$

where $\mathcal{G}\subseteq SU(N)$ has dimension $\mathrm{Dim}[\mathcal{G}] = \frac{\mathcal{K}_1}{\mathcal{K}_2}\,(\mathcal{K}_2^2\,-\,1)$. We have explicitly checked up to the case of $N = 16$ ($SU(16)$) that the symmetry breaking is compatible with

$$SU(N)\quad\rightarrow\quad SU(\mathcal{K}_2)^{\frac{\mathcal{K}_1}{\mathcal{K}_2}}\,\times\,U(1)^{\frac{\mathcal{K}_1}{\mathcal{K}_2}-1}\,. \tag{C.14}$$

Since this result depends only on quantities invariant under the transformations in eq. (C.2), as $\mathcal{K}_1$ and $\mathcal{K}_2$, we can finally extend such result to all the matrices of the general form of eq. (C.1).

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